Editors-in-Chief
Aliev Fikret (Azerbaijan)
Başar Tamer (USA)

Deputy Editor-in-Chief
Safarova Nargiz (Azerbaijan)

Editorial Board
Aidazadeh Kamil (Azerbaijan)          Lakestani Mehrdad (Iran)
Akbarov Surkhay (Turkey)              Larin Vladimir (Ukraine)
Bagirov Adil (Australia)              Mahmudov Nazim (TRNC)
Colaneri Patrizio (Italy)             Ozbay Hitay (Turkey)
Hajiyev Asaf (Azerbaijan)             Patrick Sole (France)
Kabanikhin Sergey (Russia)            Pogorilyy Sergey (Ukraine)
Konstantinov Mihail (Bulgaria)        Ruzhansky Michael (UK)

Executive Editors
Mammadova Gamar (Azerbaijan)
Rusatrmova Lamiya (Azerbaijan)

Editorial Assistants
Hajiyeva Nazile (Azerbaijan)
Huseynova Nargiz (Azerbaijan)

PROCEEDINGS

of the

6th International Conference on

CONTROL AND OPTIMIZATION
WITH INDUSTRIAL APPLICATIONS

Volume I

July 11-13, 2018
Baku, Azerbaijan
Proceedings of the 6th International Conference on Control and Optimization with Industrial Applications (COIA 2018)

PREFACE

This volume of Proceedings contains selected papers from the 6th International Conference on Control and Optimization with Industrial Applications (COIA 2018) held in Baku, Azerbaijan, on July 11-13, 2018. The conference, which was organized by the Ministry of Transport, Communications and High Technologies of the Republic of Azerbaijan, and the Institute of Applied Mathematics of Baku State University, has received more than 300 abstracts. Following a review process, 222 of these were accepted for presentation at the conference. We thank all participants for their contributions to the Conference program and for their contributions to these Proceedings.

The topics that are covered in the conference include Control Theory, Optimization, Intelligent Systems, Fuzzy Control, Numerical and Computational Methods, Network and Telecommunications, Mathematical Modelling and Simulation, Applications in Industrial Processes and Economics, and Identification.

Reviewing and evaluating the submitted abstracts to COIA 2018 was a challenging undertaking that relied on the goodwill of many researchers who are experts in the topical areas covered by the conference. More than 65 researchers were involved in that process, and we thank them for their time and effort in reviewing the submissions and in providing useful feedback to the authors, which the final versions of the papers included in these Proceedings have benefited from.

We would like to express our deep appreciation to the conference sponsor, Ministry of Transport, Communications and High Technologies of the Republic of Azerbaijan for their financial support.

As this conference is being closed, we look forward to the next one in the series, the 7th International Conference on Control and Optimization with Industrial Applications, which will be held in 2020.

Aliev Fikret
Tamer Başar
CONTENTS

Dynamic parameter adaptation using type-2 fuzzy logic in bio-inspired optimization methods for fuzzy control ........................................ Castillo O. 13

Method of splitting operators in matrix analysis and control ............................................................... Konstantinov M., Petkov P. 16


Reaction-diffusion models: dynamics and control ................................................................. Zuazua E. 22

Optimization of constants for subelliptic embeddings .............................................................. Ruzhansky M. 25

Modeling with splining ................................................................................................................ Sarfraz M. 28

Multidimensional analogs of Gelfand-Levitan-Krein equations ................................................... Kabanikhin S., Shishlenin M., Novikov N. 31

The application of fuzzy logic and multi-fractal analysis for reservoir management ..................... Suleimanov B.A., Ismailov F.S., Huseynova N.I., Veliev E.F. 34

Application of methods of data mining in the educational process ................................................ Abbasov A., Aliyeva T. 37

Second-order coexhausters calculus .......................................................................................... Abbasov M.E. 40

The initial concept of the new theory for the mathematical modeling of turbulent flows .................. Akhundov A.A., Akhundova E.M. 43

Some issues of application of internet of things in the oil and gas complex .................................. Algulyev R., Fataliyev T., Mehdiyev Sh. 46

Convergence and stability of new iterative scheme in Banach space and application ..................... Ali J., Ali F. 49

Transformation of the Mittag-Leffler function to an exponential function and its some applications to problems with a fractional derivative ......................................................... Aliyev F.A., Aliyev N.A., Safarova N.A., Gasimova K.G. 52
Asymptotical method to solution of identification problem for defining the parameters of discrete dynamical system in gas-lift process ......................... Aliev F.A., Hajiyeva N.S., Namazov A.A., Safarova N.A., Huseynova N.Sh.  55

On the solving of Sylvester type matrix equation ......................... Aliev F.A., Larin V.B.  58


Limit theorems for the family of the first passage time of the parabola by a random walk described by the autoregression process of order one (AR(1)) ......................... Aliev S.A., Rahimov F.H., Hashimova T.E., Farhodova A.D., Bagirova G.A.  65

Light-cone distribution amplitudes of light $J^{PC} = 2^{--}$ tensor mesons in QCD ......................... Aliev T.M., Bilmis S., Yang K.Ch.  68

An investigation of a mixed problem for the parabolic equation second-order ......................... Aliev A.M., Aliev N.A  71

On a coupled system of fractional differential equations with integral boundary conditions ......................... Alkhateeb A.  74

Application of fuzzy logic for risk determination of type 2 diabetes disease ......................... Allahverdi N., Ertosun N.  77

A resent survey on numerical methods for solving singularly perturbed problems ......................... Amirali I., Amiraliyev G.M.  80

On the boundaries of changing parameters in the mathematical modeling of the dynamic system of the figure of the earth ......................... Arazov G.T., Aliyeva T.H.  83

Discrete optimal control with closed loop poles in a circular regions in frequency domain ......................... Arcasoy C.C., Eroglu H.  86

About generalized solutions of basic boundary value problems for the second order elliptic equation in unbounded domains ......................... Asadzadeh T.B.  89

Convergence of HP-streamline diffusion and Nitsche’s schemes for the relativistic Vlasov-Maxwell system ......................... Asadzadeh M., Kowalczyk P., Standar C.  92

Biorthogonal multiwavelets on the interval for solving multi-dimensional fractional optimal control problems with inequality constraint ......................... Ashpazadeh E., Lakestani M.  95


Numerical solutions of sequential fractional equations by using fractional Taylor basis ......................... Avci I.  101

On the solution of the stability problem of three-layer systems with functionally graded (FG) interlayer in the industrial applications ......................... Avey A., Najafov A.M.  104
Numerical solution to optimal control problems for loaded dynamic systems with integral conditions .......................... Aida-Zade K.R., Abdullayev V.M. 107

Numerical solution to optimal control for wave process with the set of initial conditions ............ Aida-Zade K.R., Ashrafova Y.R. 110

Spectral properties and scattering problems of eigenparameter dependent discrete impulsive Sturm-Liouville equations ........ Aygar Y., Bairamov E., Oz bey G. 113

Haar wavelet method for numerical solution of pantograph functional differential equations ........ Aziz I., Yasmeen Sh., Amin R. 116

Numerical solution of a Volterra integral-algebraic equations .......... Babakhani A. 119

Magnetic moment of electrons in diluted magnetic semiconductor quantum ring ................ Babanli A.M., Ibragimov B.G. 122

On sequential fractional differential equations with integral boundary conditions ........ Bawaneh S. 125


Investigation of a discrete Dirac system with an interaction point ................ Cebesoy S., Bairamov E., Solmaz S. 131

Covariance switching in vibration control .................. Colaneri P. 134

Radial position-momentum uncertainties for the infinite spherical well and the fisher entropy ........ Dong Q., Torres-Arenas A.J., Sun G.H., Dong S.-H. 137

Controlled queues with correlated arrival process and reserve servers for solving energy saving problems in cloud computing systems .................. Dudin A.N., Dudin S.A., Klimenok V.I., Dudina O.S. 140

Sensitivity analysis of one optimization problem with prehistory ........ Dvalishvili Ph. 143

Stable plasma structures as an alternative energy source .................. Dvornikov M., Jafarov R.G., Mutallimov M.M., Valizade A.H. 146

Inverse spectral problem for Hill operator on lasso graph .......... Efendiev R. 149

The optimal problem related to change in the body shape ................ Efendiyeva H.J., Rustamova L.A. 152

Fractional differential equations with nonlocal Katugampola fractional integral ................ Emin S. 155

Edge-neighbor-rupture degree of graphs and examined on thorny graph ................ Eskiizmirliler S., Yorgancioglu Z.O., Polat R., Gursoy M.U. 158

Solving free boundary problem for an initial cell layer in multispecies biofilm formation by Newton-Raphson method .................. Fazlallah M.A., Ivaz K. 161
The problem of optimal synthesis of the 3D-multidimensional nonlinear modular dynamic systems. Feyziev F.G., Mammadova G.H., Mekhtieva M.R. 164

Parametric identificational determination of the functions of relative phase permeabilities of gas-condensate deposits in water displacement mode. Feyzullayev Kh.A., Khalilov M.S., Kutiev E.A. 167

The problem of identification of the kinetic coefficient and the right part in convection-reaction equation. Gamzaev Kh.M. 170

Optimal control of the quasi-linear neutral differential equation. Gorgodze N. 173

A general and suitable metrics in fuzzy space. Gurbanov F.I., Mamedova N.G. 176

Optimization of mechanized methods of operation taking into account the production gas rate. Gurbanova T.G., Gurbanov R.S. 179

On the boundary functional of the random walk with two barriers related to optimal capacity of the buffer stock. Hanalioglu Z., Gever B., Poladova A., Khaniyev T. 182

Estimation of impact of innovations on the quality of tertiary education. Hasanli Y.H., Shabanov S.A. 185

Mathematical model of mass-exchange in medium fractal structures. Hasanli A.B., Sadigov E.N. 188

Numerical algorithms for solving the inverse problem. Huseynova N.Sh., Orucova M.Sh., Safarova N.A., Hajiyeva N.S. 191

Definition and classification of variables that form evaluation method for drive mechanism decision making guide for in-pipe inspection robots in oil pipelines. Ibrahimov B., Alili A. 194

Finite size and topological effect in gauged four-fermion interaction model. Inagaki T. 198

Development of the combined algorithm for increasing the measurement accuracy. Isayev M.M. 201

Summation the statistical indicators and forming the database for wells covering the same layer. Ismailov N.A., Aliev F.A., Maharramov I.A., Gasimova K.G. 204

On extremality of some algebraic varieties. Jabbarov I.Sh., Hasanova G.K. 206

Investigation and solving some fractional partial differential equations by spectral and contour integral methods. Jahanshahi F., Jahanshahi M. 209

Spectral problem for an initial-boundary value problem involving first order two dimensional generalized nonhomogenous Cauchy-Riemann equation with general non-local boundary conditions. Jahanshahi M., Ebadpour Golanbar J., Aliyev N. 212
Approximate solutions for some nonlinear fractional partial differential equations by Adomian decomposition method and Mittag-Leffler functions ........................................... Jahanshahi M., Demneh H.K. 216

Linear seismic data processing of area observing systems ........................................... Kabanikhin S.I., Novikov N.S., Shishlenin M.A. 219

New formulations for the traveling repairman problem with time windows .......................... Kara I., Onder Uzun G. 222

Augmented Lagrangian based solution method for one-dimensional cutting stock and assortment problem ........................................... Kasimbeyli N., Sarac T., Kasimbeyli R. 225

Optimal control of a giving up smoking model with age-structured in smoking classes ........ Khan A., Zaman G. 228

On the solution of one problem for linear hyperbolic type loaded differential equation by the method of finite differences ........................................... Khankishiyev Z.F. 231

Some inconsistencies of familiar quantum mechanical relations in case of singular potentials and operators ........................................... Khelashvili A., Nadareishvili T. 234

Principal functions of discrete Sturm-Liouville equation with generalized eigenparameter in boundary condition ........................................... Koprubasi T. 237


Optimization of mayer problem with differential inclusions and polynomial differential operators ........................................... Mahmudov E.N. 243

Finite-approximate controllability of semilinear evolution systems via resolvent-like operators ........................................... Mahmudov N.I. 246

3D optimal control problem for a Manjeron generalized equation with non-classical Goursat conditions ........................................... Mamedov I.G. 249

Contributions of magnetic type interactions to the vector meson-nucleon coupling in the bottom-up approach ........................................... Mamedov Sh., Taghiyeva Sh. 252

Finding the guaranteed suboptimal solution to the functional in the integer programming problem ........................................... Mammadov K.Sh., Mammadov N.N. 255

Methods for evaluation of human resources performance in virtual organizations ................ Mammadova M.H., Jabrayilova Z.G. 258

Multipoint necessary optimality conditions of singular controls in delayed stochastic systems ........................................... Mansimov K.B., Mustaliyev R.O. 261

To sufficient conditions for strong extremum in calculus of variations ................................ Mardanov M., Melikov T., Malik S. 264
Trapezoidal quadrature rule for solving nonlinear fuzzy fractional integro-differential equations of the Hammerstein type. 

...Mashhadi Gholam A., Ezzati R., Allahvaranloo T.  267

Parametric resonance of the system consisting of the circular hollow cylinder and surrounding elastic medium under action in the interior. 

...Mehdiyev M.A., Akbarov S.D.  270

A hybrid advanced multistep method for solving ODE. 

...Mehdiyeva G.YU., Ibrahimov V.R., Imanova M.N.  273

Method of the reserve resources determination for the distributed computer systems with the network-centric resource control. 


Crude oil price forecasting techniques in the world market. 

...Muradov A., Hasanli Y., Hajiyev N.  279

Boundary value problem for the anisotropic type convolution equations. 

...Musaev H.K.  282

Necessary conditions for the Fredholmness of three-dimensional Helmholtz equation with non-local boundary value conditions. 

...Mustafayeva Y.Y., Aliyev N.A.  285

New sweep algorithm for solving optimal control problem with multi-point boundary conditions. 

...Mutallimov M.M., Amirova L.I., Aliyev F.A., Faradjova Sh.A.  288

Optimization of the groove cutting condition by the vortex methods on the rotational surface. 

...Nadirov U.  291

On a time evolution of the quadratic quantum systems. II. 

...Nagiyev Sh.M., Ahmadov A.I., Amirova Sh.A.  294

Spectral theory of Hilfer fractional Bessel operator. 

...Panakhov E., Bas E., Ercan A.  297

Mathematical model for calculating the dynamic seal assembly and the problem of optimization of its parameters. 

...Pashayev A.B., Sabziev E.N.  300

A formalized approach to verify GPGPU applications. Part 1. 

...Pogorilyy S.D., Kryvyi S.I., Slynko M.S.  303

Formalized approach to verify GPGPU applications. Part 2. 

...Pogorilyy S.D., Kryvyi S.I., Slynko M.S.  306

An approach to solution to class of coefficient-inverse nonlocal problems for a parabolic equation. 

...Rahimov A.B.  309

On stability of the gradient algorithm for one separable nonlinear discrete optimization problems. 

...Ramazanova E.E., Gurbanov R.S., Mammadova M.A., Malikov H.Kh., Hajiyev A.A.  312

Control and optimization of fluid motion in microcrack channels by means of industrial applications. 

...Ramazanova E.E., Gurbanov R.S., Mammadova M.A., Malikov H.Kh., Hajiyev A.A.  315
On asymptotics of the solution of boundary value problem for quasi-linear elliptic equation in curvilinear trapezoid. Sabzaliev M.M., Sabzalieva I.M. 318

A gradient in optimal control problem of processes described by first order system. Sadygov M.A., Mirzayeva H.G., Akhundov H.S. 321

Subdifferential of the second order and condition of the optimality. Sadygov M.A. 324

Experiment of solar energy plants application in thermal effect process of the oil reservoir. Salavatov T., Mammadov F. 327

The metallicity of atmosphere of the post AGB HD161796 (F3Ib) star. Samedov Z.A. 330

An approach to improve remote photoplethysmography signal quality. Samet R., Yaman A.U. 333

Real-time video synopsis for surveillance cameras. Samet R., Baskurt K.B. 336

Modeling curves by a quadratic trigonometric B-spline with point shape control. Sarfraz M., Samreen S., Hussain M.Z. 340

On the inverse spectral problem for Sturm-Liouville operator having special singularity. Sat M., Panakhov E.S., Kayalar M. 343

Impacts of the devaluation on Azerbaijan export. Sharifzadeh E.R., Aliyev R.M., Bayramov V.A., Huseynov J.P. 346

Optimization methods in the stability investigation of regulator systems. Shatyrko A.V., Khusainov D.Ya. 349

Variation formulas for delay differential equations and necessary optimality conditions. Shavadze T. 352

Fuzzy approach to determine the thickness of the oil slick on the water surface. Shikhlinskaya R.Y., Ahmadova R.Y., Mirzayev F.A. 355

Local meshless method for convection dominated steady and unsteady partial differential equations. Siraj I.U. 358

A new deep learning based system for facial gesture analysis. Soylu B.E., Guzel M.S., Askerzade I.N. 361

Optical solitons to the conformable time-fractional perturbed Radhakrishnan-Kundu-Lakshmanan equation. Sulaiman A.T., Bulut H., Yel G., Atas S.S. 364

Sensitivity analysis of delay differential equations and optimization problems. Tadumadze T. 367

On topologically uniformly transitive actions of locally compact unimodular amenable groups. Tagi-Zadeh A.T. 370
The frequency high accuracy algorithm for the solution of continuous linear-quadratic optimal synthesis problem by output variable ........................................ Velieva N.I. 373

The Goursat problem for the pseudoparabolic equation of the third order with singular coefficients .................................................. Yagubov M.H., Yusubov Sh.Sh. 376

On the design of internal model generators for repetitive controllers .................................................. Yucesoy V., Ozbay H. 379
DYNAMIC PARAMETER ADAPTATION USING TYPE-2 FUZZY LOGIC IN BIO-INSPIRED OPTIMIZATION METHODS FOR FUZZY CONTROL

OSCAR CASTILLO

1Tijuana Institute of Technology, Calzada Tecnologico Tomas Aquino, Mexico
e-mail: ocastillo@tectijuana.mx

Abstract. In this paper we perform a comparison of the use of type-2 fuzzy logic in two bio-inspired methods: Ant Colony Optimization (ACO) and Gravitational Search Algorithm (GSA). Each of these methods is enhanced with a methodology for parameter adaptation using interval type-2 fuzzy logic, where based on some metrics about the algorithm, like the percentage of iterations elapsed or the diversity of the population, we aim at controlling their behavior and therefore control their abilities to perform a global or a local search. To test these methods two benchmark control problems were used in which a fuzzy controller is optimized to minimize the error in the simulation with nonlinear complex plants.

Keywords: Interval type-2 fuzzy logic, ant colony optimization, gravitational search algorithm, dynamic parameter adaptation.

AMS Subject Classification: 68T01, 93C42.

1. Introduction

Bio-inspired optimization algorithms can be applied to most combinatorial and continuous optimization problems, but for different problems need different parameter values, in order to obtain better results. There are in the literature, several methods aim at modeling better the behavior of these algorithms by adapting some of their parameters [16], introducing different parameters in the equations of the algorithms [4], performing a hybridization with other algorithms, and using fuzzy logic [5-9, 14]. In this paper a methodology for parameter adaptation using an interval type-2 fuzzy system is presented, where on each method a better model of the behavior is used in order to obtain better quality results.

The proposed methodology has been previously successfully applied to different bio-inspired optimization methods like BCO (Bee Colony Optimization) in [1], CSA (Cuckoo Search Algorithm) in [3], PSO (Particle Swarm optimization) in [5, 7], ACO (Ant Colony Optimization) in [6, 8], GSA (Gravitational Search Algorithm) in [9, 16], DE (Differential Evolution) in [10], HSA (Harmony Search Algorithm) in [11], BA (bat Algorithm) in [12] and in FA (Firefly Algorithm) in [15].

The algorithms used in this research are ACO (Ant Colony Optimization) from [8] and GSA (Gravitational Search Algorithm) from [9], each one with dynamic parameter adaptation using an interval type-2 fuzzy system. Fuzzy logic help us to model a complex problem, with the use of membership functions and fuzzy rules, with the knowledge of a problem from an expert, fuzzy logic can bring tools to create a model and attack a complex problem. The contribution of this paper is the comparison between the bio-inspired methods which use an interval type-2 fuzzy system for dynamic parameter adaptation, in the optimization of fuzzy controllers for nonlinear
complex plants. The adaptation of parameters with fuzzy logic helps to perform a better design of the fuzzy controllers, based on the results which are better than the original algorithms.

The proposed methodology for parameter adaptation is illustrated in Fig. 1, where it has the optimization method, which has an interval type-2 fuzzy system for parameter adaptation.

![Figure 1. General scheme of the proposal for parameter adaptation.](image)

The optimization of a fuzzy controller is a complex task, because require the search of several parameters in infinite possibilities in the range of each input or output variables. The bio-inspired optimization methods help in the search because is guided by some kind of intelligence, from swarm intelligence or from laws of physics and can make a better search of parameters. With the inclusion of a fuzzy system in this case an interval type-2, the bio-inspired methods can search even in a better way, because is guided by the knowledge of an expert system that model a proper behavior in determined states of the search, in the beginning improves the global search or exploration of the search space and in final improves the local search or the exploitation of the best area found so far of the entire search space. From the results with the MSE there is clearly that ACO with parameter adaptation has the best results in the robot problem, and GSA with parameter adaptation has the best results in the shower problem, but with the statistical test it confirm these affirmations. The statistical comparison shows that the methods with parameter adaptation are better than their counterparts the original methods. Also ACO is a better method with the robot problem, but GSA is better in the shower problem.

As future work, other meta-heuristics can be considered for fuzzy parameter adaptation, like others mentioned in [17-19]. Fuzzy logic [20-22] can also be considered in other operations or representations of meta-heuristics, which can also be interesting to explore.

2. Conclusion

Type-2 fuzzy logic can be used to enhance bio-inspired optimization methods by providing them with dynamic parameter adaptation. Two particular methods are considered to test the proposed approach, namely ACO and GSA. The type-2 fuzzy versions of ACO and GSA are more efficient than the original methods. The methods were used to optimize the design of fuzzy controllers in two complex plants. In particular, the design of a fuzzy controller of an autonomous mobile robot is considered with good results.

As future work, other meta-heuristics can be considered for fuzzy parameter adaptation, like others mentioned in [17-19]. Fuzzy logic [20-22] can also be considered in other operations or representations of meta-heuristics, which can also be interesting to explore.
References


Method of splitting operators (MSO). We consider the powerful MSO and its main applications to the perturbation analysis of problems involving unitary transformations in matrix theory and control. A brief survey of results in this area is also presented.

Let \( C^{m \times n} \) and \( F_n \subseteq C^{m \times n} \) be the sets of complex \( m \times n \) and upper triangular \( n \times n \) matrices, resp. A matrix operator \( P : C^{m \times n} \rightarrow C^{m \times n} \) and a matrix argument \( X \in C^{m \times n} \) are split into strictly lower, diagonal and strictly upper parts as \( X = X_1 + X_2 + X_3 \) and \( P = P_1 + P_2 + P_3 \), where \( X_k = L_k(X) \) and \( P_k = L_k \circ P \). The matrices \( X_k \) contain many zeros at fixed positions so we work with their compressed vector representations \( l_k(X) \), where \( l_1(X) \in C^{n(n-1)/2} \) for \( X \in C^{m \times n} \), etc. Explicit expressions for the matrices of the operators \( l_k \) are straightforward, see e.g. \([7,9,11]\). If \( T \in F_n \) and \( X \in C^{m \times n} \) then the matrices \( L_1(XT) \) and \( L_1(TX) \) depend only on \( L_1(X) \) rather than on the whole matrix \( X \). More generally, if \( L : C^{m \times n} \rightarrow C^{m \times n} \) is determined from \( L(X) = AXB \), then \( L_1 \circ L = L_1 \circ K \circ L_1 \). This is a fundamental property in using MSO.

We consider matrix decompositions \( R = \Psi(A, U) \), where \( A \) is a matrix (or collection of matrices), the matrix \( R \) is upper triangular and the matrix \( U \) is unitary. If \( A + E \) is a perturbation of \( A \) and \( R + Z = \Psi(A + E, U + V) \) is the solution of the perturbed problem then MSO allows to find estimates for the norms of \( Z \) and \( V \) as functions of the norm of \( E \).

QR decomposition (QRD). The QRD \( A = UR \) of a matrix \( A \in C^{m \times n} \), where the matrix \( U \) is unitary and \( R \) is upper triangular is a major tool in matrix computations such as rank determination and solution of linear equations and least square problems. Explicit estimates for \( \|V\| \) and \( \|Z\| \) as functions of \( \|E\| \) are derived by MSO, where \( R = U^H A \) and \( A + E = (U + V)(R + Z) \).

Singular value decomposition (SVD). Similar results are obtained for the SVD \( A = U_1 R U_2^H \), where the matrices \( U_1, U_2 \) are unitary and \( R \) is diagonal with the singular values of \( A \) on its diagonal.

Schur decomposition (SD). The perturbation analysis of SD \( A = U R U^H \) of the matrix \( A \in C^{n \times n} \) is done again by the MSO under the assumption that \( A \) has pairwise disjoint eigenvalues. Note that if there are multiple eigenvalues then the Schur form \( R \) may not be Lipschitz continuous relative to perturbations in \( A \).

Hamiltonian-Schur decomposition (HSD). In the HSD we have \( A = [A_1, A_2; A_3, -A_1^H] \) where the MATLAB notation for block matrices is used and \( A_2^H = A_2, A_3^H = A_3 \). The transformation matrix \( U = [U_1, U_2; -U_2, U_1] \) is unitary symplectic, while the condensed form \( R = U^H A U = [R_1, R_2; 0, -R_1^H] \) satisfies \( L_1(R_1) = 0, R_2^H = R_2 \). The less condensed form 2-block Schur form \( \tilde{R} = [\tilde{R}_1, \tilde{R}_2; 0, \tilde{R}_3] \) is also analyzed. A 3-block Schur form is also analyzed by MSO and the results are to be published in the full version of the paper.
Unitary canonical forms (UCF). The sensitivity of UCF \([A^0, B^0] = [U^H A U, U^H B]\) of linear control systems is effectively analyzed by MSO. The sensitivity of UCF with \(B^0 = U^H B W\), where \(W\) is unitary is also studied.

Modal Control (MC). The general problem of MC (pole assignment synthesis in particular) of linear systems \(x' = Ax + BU, y = Cx\) is to find a feedback matrix \(K\) such that the closed-loop system matrix \(A + BKC\) to be unitary similar to a given (attainable by output feedback) matrix \(A_c\). When the triple \((C, A, B)\) and the matrix \(A_c\) are perturbed one has to estimate the perturbation in \(K\). This problem is again solved effectively by MSO.

Survey of results. Perturbation analysis in matrix analysis and control is of major interest for theory and computations [4,7–10,14], see also [3,23]. Problems involving unitary transformations are particularly important since modern numerical methods and algorithms are based on such transformations. The MSO was firstly proposed in [17] in implicit form. The method has been further developed in [7–11,14–16]. The application of MSO to the perturbation analysis of the QR decomposition is presented in [1,7,8,24]. The perturbation analysis of SVD by MSO is outlined in [10].

The sensitivity of the Schur system of a matrix has been one of the first rigorous implementations of MSO [11]. The generalized Schur decomposition has been studied in [25]. A perturbation analysis technique for the Hamiltonian-Schur forms of matrices arising in optimal control is presented in [6]. The application of MSO to the sensitivity analysis of unitary (orthogonal in particular) canonical forms of controllable systems is considered in [9,13–16]. MSO is effectively applied to the sensitivity analysis of the general problem of feedback synthesis of linear control systems (pole assignment synthesis in particular) in [12,16,18,27]. Other applications of MSO are presented in [2].

Alternative and effective approach to the perturbation analysis of unitary (orthogonal) matrix factorizations is given in [19–22]. A general perturbation theory of linear operators is given in the classics [5], while a rigorous study of matrix perturbation problems is presented in [23].

Keywords: Perturbation analysis, splitting operators, unitary transformations, matrix decompositions.

AMS Subject Classification: 47A55, 65F25, 93B10, 93B52.

References

TECHNOLOGIES AND SYSTEMS OF NOISE CONTROL OF THE BEGINNING AND DYNAMICS OF DEVELOPMENT OF ACCIDENTS AND THEIR APPLICATION IN OIL AND GAS PRODUCTION AND CONSTRUCTION

T.A. ALIEV¹, O.G. NUSRATOV¹, N.F. MUSAeva², G.A. GULUEV¹, A.G. RZAEV¹, F.H. PASHAYEV¹, U.E. SATTAROVA², T.A. ALIZADA¹, N.E. RZAYEVA²

¹Institute of Control Systems of Azerbaijan National Academy of Sciences, Baku, Azerbaijan
²Azerbaijan University of Architecture and Construction, Baku, Azerbaijan
e-mail: director@cyber.az

1. Introduction

It is shown in the paper that the beginning of the latent period of transition of most facilities into an emergency state on the basis of the results of traditional technologies is registered with a delay because of the emergence of noises correlated with the useful signal, which in some cases causes accidents with catastrophic consequences. New technologies and systems of noise control of the beginning and the development dynamics of accidents are proposed. Examples of the principles for constructing various noise control intelligent systems, the possibility of their implementation at oil and gas production facilities, drilling rigs [1], offshore fixed platforms [2] and compressor stations [3], in transport, aviation, power engineering, construction, seismology [4] and medicine [5] are given. It is also shown that the use of the noise technologies can improve the accuracy of the results of traditional methods for analyzing noisy random signals.

2. Computation of characteristics of the noisy signal

In real different-purpose control and management systems, under normal technical condition of facilities, the noisy signals \( g(t) \) obtained at the sensor outputs are the sum of the useful signal \( X(t) \) and the noise \( \varepsilon_1(t) \):

\[
g(t) = X(t) + \varepsilon_1(t).\]

In this case, the known classical conditions are fulfilled for the centered noisy signals \( g(t) \):

\[
\begin{align*}
M[X(t)X(t)] &\neq 0, M[\varepsilon_1(t)X(t)] = 0, \\
M[X(t)\varepsilon_1(t)] &= 0, M[\varepsilon_1(t)\varepsilon_1(t)] \neq 0.
\end{align*}
\]

As a result, the formula for calculating the variance of the signal \( g(t) \) takes the form

\[
D_{gg} = M[g(t)g(t)] = M[(X(t) + \varepsilon_1(t))(X(t) + \varepsilon_1(t))]
= M[X(t)X(t) + X(t)\varepsilon_1(t) + \varepsilon_1(t)X(t) + \varepsilon_1(t)\varepsilon_1(t)]
= M[X(t)X(t) + \varepsilon_1(t)\varepsilon_1(t)].
\]

Therefore, we have

\[
D_{gg} = M[X(t)X(t) + \varepsilon_1(t)\varepsilon_1(t)] = D_{XX}(0) + D_{\varepsilon},
\]

where

\[
D_{\varepsilon_1} = M[\varepsilon_1(t)\varepsilon_1(t)] = M[\varepsilon(t)\varepsilon(t)] = D_\varepsilon.
\]
The process of accident initiation manifests itself in the signal \( g(t) \) as the noise \( \varepsilon(t) \). For this reason, starting from the moment of initiation and development of the latent period of an accident, the model of the signal \( g(t) \) can be represented as

\[
g(t) = X(t) + \varepsilon_1(t) + \varepsilon_2(t).
\]

Here the presence of a correlation between the useful signal \( X(t) \) and the sum noise

\[
\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t)
\]

leads to the following inequalities

\[
M[X(t)X(t)] \neq 0 \quad (1)
\]
\[
M[X(t)\varepsilon_2(t)] \neq 0 \quad (2)
\]
\[
M[\varepsilon_1(t)\varepsilon_1(t)] \neq 0 \quad (3)
\]
\[
\varepsilon(t)\varepsilon(t) \neq 0 \quad (4)
\]
\[
\varepsilon(t)X(t) \neq 0 \quad (5)
\]

and equalities

\[
M[\varepsilon_1(t)X(t)] = 0, \varepsilon_1(t)\varepsilon_2(t) = 0.
\]

Therefore, we have

\[
D_{gg} = M\{[X(t) + \varepsilon_1(t) + \varepsilon_2(t)][X(t) + \varepsilon_1(t) + \varepsilon_2(t)]\}
\]
\[
= M[X(t)X(t) + X(t)\varepsilon_1(t) + X(t)\varepsilon_2(t) + \varepsilon_1(t)X(t) + \varepsilon_1(t)\varepsilon_1(t)
\]
\[
+ \varepsilon_1(t)\varepsilon_2(t) + \varepsilon_2(t)X(t) + \varepsilon_2(t)\varepsilon_1(t) + \varepsilon_2(t)\varepsilon_2(t)]
\]
\[
= M[X(t)X(t) + \varepsilon_2(t)X(t) + X(t)\varepsilon_1(t) + \varepsilon_1(t)\varepsilon_1(t) + \varepsilon_2(t)\varepsilon_1(t) + \varepsilon_2(t)\varepsilon_2(t)]
\]
\[
= R_{XX}(t) + 2R_{X\varepsilon_2}(t) + D_{\varepsilon_1\varepsilon_1} + D_{\varepsilon_2\varepsilon_2},
\]

where

\[
D_{\varepsilon} = M[\varepsilon_2(t)X(t) + X(t)\varepsilon_2(t) + \varepsilon_1(t)\varepsilon_1(t) + \varepsilon_2(t)\varepsilon_2(t)]
\]
\[
= 2R_{X\varepsilon_2}(0) + D_{\varepsilon_1\varepsilon_1} + D_{\varepsilon_2\varepsilon_2} = 2R_{X\varepsilon} + D_{\varepsilon\varepsilon},
\]
\[
D_{\varepsilon\varepsilon} = D_{\varepsilon_1} + D_{\varepsilon_2}.
\]

3. Technologies and systems of noise control

Due to this, when traditional signal analysis technologies are used, an error emerges and as a result, the beginning of an accident is registered with a delay. For this reason, the following technologies for controlling the beginning of the latent period of an accident have been developed.

(1) Technology for adaptive determination of the sampling interval of the noise of the noisy signal.

(2) Technology for calculating the variance of the noise of the noisy signal.

(3) Technology for forming the noise equivalent to the samples of the sum noise of the noisy signal.

(4) Technology for forming the samples of analogs of the correlated noise of the noisy signals that arise in the latent period of an emergency state.

(5) Technology for spectral analysis of the sum noise of the noisy signal both in the presence of a correlation between the noise and the useful signal and in the absence of such.

(6) Technology for spectral analysis of the equivalent correlated noise of the noisy signal.

(7) Technology for calculating the cross-correlation function between the useful signal and the noise.

(8) Technology for forming the correlation matrices equivalent to the latent period of control objects emergency state.
(9) Technology for forming the normalized correlation matrices equivalent to the matrices of the useful signals.

(10) Correlation noise control technology.

(11) Spectral noise control technology.

(12) Position-binary noise control technology.

(13) Correlation relay noise control technology.

(14) Spectral relay noise control technology.

(15) Technologies for correlation and spectral noise control of accident development dynamics.

Based on them, the principles of building the following noise control systems have been proposed.

(1) System of noise control of the beginning of the latent period of accidents on fixed offshore platforms.

(2) Technologies and system of noise control of the beginning and development dynamics of accidents on drilling rigs.

(3) System of noise control of sucker rod pumping units.

(4) System of noise monitoring of the beginning of the latent period of accidents on compressors stations.

(5) Technologies and system of noise control of anomalous seismic processes.

(6) Digital citywide system of noise control of the technical condition of socially significant objects.

(7) Intelligent seismic-acoustic system for identifying the area of the focus of an expected earthquake.

(8) Possibilities of using laptops and smartphones for the control of the state of the heart.

(9) Technology for monitoring the state of the heart based on the spectral characteristics of heart sounds.

(10) Correlation system for monitoring the beginning of the latent period of vascular pathology of the human body.

4. Conclusion

The use of these intelligent systems of noise control of the beginning of the latent period and the development dynamics accidents on real-life technical facilities has shown that they allow increasing the level of safety and reliability of their operation.

Keywords: Control, technical condition, accident, signal, noise, technology.

AMS Subject Classification: 93C83.

References


REACTION-DIFFUSION MODELS: DYNAMICS AND CONTROL

ENRIQUE ZUAZUA\textsuperscript{1,2,3,4}

\textsuperscript{1}DeustoTech, University of Deusto, Basque Country, Spain
\textsuperscript{2}Departamento de Matemáticas, Universidad Autónoma de Madrid, Madrid, Spain
\textsuperscript{3}Facultad Ingeniería, Universidad de Deusto, Avda Universidades, Spain
\textsuperscript{4}Sorbonne Universités, Laboratoire Jacques-Louis Lions, Paris, France

e-mail: enrique.zuazua@deusto.es

Abstract. Reaction-diffusion equations are ubiquitous and its applications include combustion and population dynamics modelling. There is an extensive mathematical literature addressing the analysis of steady state solutions, traveling waves, and their stability, among other properties. Control problems arise in many applications involving these models. And, often times, they involve control and/or state constraints, as intrinsic requirements of the processes under consideration.

In this lecture we shall present the recent work of our team on the Fisher-KPP and Allen-Cahn or bistable model. We show that these systems can be controlled fulfilling the natural constraints if time is large enough. This is in contrast with the unconstrained case where parabolic systems can be controlled in an arbitrarily small time, thanks to the infinite velocity of propagation. The method of proof combines various methods and, in particular, employs phase-plane analysis techniques allowing to build paths of steady-state solutions. The control strategy consists then in building trajectories of the time-evolving system in the vicinity of those paths. We shall conclude our lecture with a number of challenging open problems.

Keywords: Reaction-diffusion models, population dynamics, control, traveling waves, turnpike property.

AMS Subject Classification: 35K57, 35C07, 93B05, 93C20.

1. Problem formulation and main results

The Allee threshold of an ecological system distinguishes the sign of population growth either towards extinction or to carrying capacity. In practice human interventions can tune the Allee threshold for instance thanks to the sterile male technique and the mating disruption.

In this lecture we address various control problems for a system described by a diffusion-reaction equation regulating the Allee threshold, viewed as a real parameter determining the unstable equilibrium of the bistable nonlinear reaction term.

We first show that this system is the mean field limit of an interacting system of particles in which the individual behavior is driven by stochastic laws. Numerical simulations of the stochastic process show that the propagation of population is governed by traveling wave solutions of the macroscopic reaction-diffusion system, which model the fact that solutions, in bounded space domains, reach asymptotically an equilibrium configuration.
An optimal control problem for the macroscopic model is then introduced with the objective of steering the system to a target traveling wave.

![Figure 1](image1.png)

**Figure 1.** Reaction term $f(y)$ (left); traveling wave (right).

Using well known analytical results and stability properties of traveling waves, we show that well-chosen piecewise constant controls allow to reach the target approximately in sufficiently long time. We then develop a direct computational method and show its efficiency for computing such controls in various numerical simulations.

![Figure 2](image2.png)

**Figure 2.** Left and middle: state evolution of the controlled system. Right: state evolution of the uncontrolled system.

On the other hand, the Allee parameter being fixed, we consider the problem of controlling the system by means of Dirichlet controls taking their values in $[0, 1]$.

We prove that the system can always be steered to invasion, while it is possible in the case of extinction if and only if the length of the interval domain is less than some threshold value, which can be computed from the data.

In the bistable case and when the length of the space-interval is subcritical, we prove that the other intermediate homogeneous steady state, though unstable for the corresponding ODE, can be reached in finite time.

The method of proof uses several ingredients:

1. First, the possibility of getting close to a very small but positive equilibrium configuration, something that, roughly, is consequence of the asymptotic stability of the null state with null Dirichlet boundary conditions.

2. One then applies a staircase control strategy, which consists in building a path of steady states, linking the small steady state to the target one, and then applying local controllability results.
The phase plane analysis of those equations is instrumental in the whole process since it is the key tool to derive the path of steady states.

Of course such a control strategy requires the control time to be large enough, something that is natural in view of the constraints imposed to the controls.

The implemented control strategy might seem not be natural. But in fact numerical experiments show that it is nearly optimal.

This first result is also of interest to highlight the intrinsic difficulty of this kind of problems. In fact extending this results to the multi-dimensional case or to the case of systems, constitutes a very challenging problem.

2. Conclusions

The problems and results we shall present raise a number of interesting questions:

1. Addressing the same problems with other models, for instance of kinetic nature.
3. Considering the same scalar reaction-diffusion model but in several space dimensions.

3. Acknowledgements

This presentation is based on joint work with Jérôme Lohac (CNRS-Nancy), Camille Pouchol and Emmanuel Trélat (LJLL-Sorbonne Univ.), Dario Pighin (UAM-Madrid) and Jiamin Zhu (Univ. Toulouse).

Our work was motivated by discussions with J.R. Uriarte from the Faculty of Economics of the University of Basque Country (UPV/EHU) who raised the problem of modelling and control of multilingualism ([1]).

References

OPTIMIZATION OF CONSTANTS FOR SUBELLIPTIC EMBEDDINGS

MICHAEL RUZHANSKY

1Imperial College London, London, UK
e-mail: m.ruzhansky@imperial.ac.uk

1. Introduction

In this talk we discuss the dependence of the best constants in Sobolev and Gagliardo-Nirenberg inequalities on the precise form of the Sobolev space norm. The analysis is carried out on general graded Lie groups, thus including the cases of $\mathbb{R}^n$, Heisenberg, and general stratified Lie groups. The Sobolev norms may be defined in terms of Rockland operators, i.e. the hypoelliptic homogeneous left-invariant differential operators on the group. The best constants are expressed in the variational form as well as in terms of the ground state solutions of the corresponding nonlinear subelliptic equations, in the spirit of Weinstein [4]. In our case, the orders of these equations can be high depending on the Sobolev space order in the Sobolev or Gagliardo-Nirenberg inequalities, or may be fractional.

2. Outline

This abstract is based on our recent papers [2] and [3], which we will discuss in the talk. The analysis is based on the theory of function spaces and related functional analysis for higher order hypoelliptic operators.

The Gagliardo-Nirenberg inequality goes back to works of Gagliardo and Nirenberg where it was shown that the inequality

$$
\int_{\mathbb{R}^n} |u|^q dx \leq C \left( \int_{\mathbb{R}^n} |\nabla u|^2 dx \right)^{\frac{n(q-2)}{4}} \left( \int_{\mathbb{R}^n} |u|^2 dx \right)^{\frac{2q-n(q-2)}{4}},
$$

(1)

holds for all $u \in H^1(\mathbb{R}^n)$. Here one can take

$$
\begin{cases}
2 \leq q \leq \infty \text{ for } n = 2; \\
2 \leq q \leq \frac{2n}{n-2} \text{ for } n \geq 3.
\end{cases}
$$

Weinstein [4] obtained an expression for the best constant in the inequality (1), relating it to the ground states (least energy solutions) of the nonlinear Schrödinger equation

$$
-\Delta u + u = |u|^{q-2} u, \quad u \in H^1(\mathbb{R}^n).
$$

(2)

On the Heisenberg group $\mathbb{H}^N$, the subelliptic Gagliardo-Nirenberg inequality takes the form
\[
\int_{\mathbb{H}^N} |u|^q dx \leq C \left( \int_{\mathbb{H}^N} |\nabla_H u|^2 dx \right)^{\frac{Q(q-2)}{4}} \left( \int_{\mathbb{H}^N} |u|^2 dx \right)^{\frac{2q-Q(q-2)}{4}},
\]  

where \(\nabla_H\) is a horizontal gradient, \(Q = 2N + 2\) is the homogeneous dimension of \(\mathbb{H}^N\), \(2 < q < 2 + \frac{Q}{N}\). The best constant for the Gagliardo-Nirenberg inequality (3) on the Heisenberg group was expressed in terms of the ground states of the subelliptic equation

\[-\triangle_H u + u = |u|^{q-2} u, \quad u \in H^1(\mathbb{H}^N),\]

where \(\triangle_H\) is the sub-Laplacian on \(\mathbb{H}^N\), and \(H^1(\mathbb{H}^N)\) is the Sobolev space on \(\mathbb{H}^N\) with the norm

\[\|u\| := \left( \int_{\mathbb{H}^N} (|\nabla_H u|^2 + |u|^2) dx \right)^{1/2}.
\]

One of the aims of this talk is to answer the following questions:

- How do the best constants in the Gagliardo-Nirenberg inequalities (1), (3) depend on the precise formula for the Sobolev norms? For example, if we replace \(\|u\|_{H^1(\mathbb{R}^n)} = \|\nabla u\|_{L^2(\mathbb{R}^n)}\) by \(\|(-\Delta)^{1/2} u\|_{L^2(\mathbb{R}^n)}\) or by the equivalent norms

\[\|(\sum_{j=1}^n |D_{x_j}|^{2m})^{1/m} u\|_{L^2(\mathbb{R}^n)},\]

and similarly for the Heisenberg group, how does it influence the best constants in (1), (3) and the nonlinear equations (2), (4)?
- What can be said about more general Gagliardo-Nirenberg inequalities? For example, when the first order Sobolev norm in (1), (3) is replaced by higher order Sobolev norms? Also, when the \(L^2\)-norms on the right hand sides in (1), (3) are replaced by appropriate \(L^p\)-norm for other values of \(p\)?

A natural setting for our analysis will be that of graded Lie groups as developed by Folland and Stein. This is the largest class of homogeneous nilpotent Lie groups admitting homogeneous hypoelliptic differential operators. These operators are called Rockland operator, after Helffer and Nourrigat’s resolution of the Rockland conjecture. Thus, our setting will include the higher order operators on \(\mathbb{R}^n\) as well as higher order hypoelliptic invariant differential operators on the Heisenberg group, on general stratified groups, and on general graded Lie groups. We also note that the Rockland operators on graded Lie groups appear naturally in the analysis of subelliptic operators on manifolds, starting with the seminal paper of Rothschild and Stein.

The starting point of our analysis is the following Gagliardo-Nirenberg inequality on the graded group:

- **(Gagliardo-Nirenberg inequality)** Let \(G\) be graded Lie group of homogeneous dimension \(Q\) and let \(R\) be a positive Rockland operator of homogeneous degree \(\nu\). Let \(a > 0, 1 < p < \frac{Q}{\nu}\) and \(p \leq q \leq \frac{pQ}{Q-\nu p}\). Then there exists a constant \(C > 0\) such that

\[
\int_G |u(x)|^q dx \leq C \left( \int_G |R^{\frac{\nu}{p}} u(x)|^p dx \right)^{\frac{Q(q-p)}{ap^2}} \left( \int_G |u(x)|^p dx \right)^{\frac{apQ-Q(q-p)}{ap^2}},
\]

holds for all \(u \in L^p_0(G)\).  

\]
Consequently, the question arises of what is the best constant $C$ in this inequality, which we may denote by $C_{G_{N,R}} = C_{G_{N,R,a,p,q}}$ since it depends on the operator $R$ as well as on the indices $a, p, q$. The related question is of the best constant in the Sobolev (embedding) inequality

$$\left( \int_G |u(x)|^q dx \right)^{\frac{p}{q}} \leq C \int_G (|R^a u(x)|^p + |u(x)|^p) dx,$$

where $u \in L^p_{a}(\mathbb{G})$.

In this talk we will show that both the Sobolev inequality (6) and the Gagliardo-Nirenberg inequality (5) are related to the following Schrödinger equation with the power nonlinearities:

$$R^a \left(|R^a u(x)|^{p-2} R^a u(x) \right) + |u(x)|^{p-2}u(x) = |u(x)|^{q-2}u(x).$$

Moreover, they are related to the variational problem

$$d = \inf_{u \in L^p_a(\mathbb{G}), J(u)=0} \mathcal{L}(u),$$

for functionals

$$\mathcal{L}(u) = \frac{1}{p} \int_G |R^a u(x)|^p dx + \frac{1}{p} \int_G |u(x)|^p dx - \frac{1}{q} \int_G |u(x)|^q dx$$

and

$$J(u) = \int_G (|R^a u(x)|^p + |u(x)|^p - |u(x)|^q) dx.$$

Keywords: Optimization, best constants, variational problems.

AMS Subject Classification: 22E30, 43A80.

References


MODELING WITH SPLINING

MUHAMMAD SARFRAZ1

1Department of Information Science, College of Computing Sciences and Engineering, Kuwait University, Safat 13060, Kuwait, e-mail: prof.m.sarfraz@gmail.com

ABSTRACT. The generation of spline curves is a useful and powerful tool in Computer Aided Geometric Design (CAGD). Although the splines have many elegant properties discussed in the current literature, the curves sometimes exhibit undesirable oscillations. Various methods have been developed to control the shape of a curve. Some methods are well suited for one type of shape control but not well suited for another. For this reason, a multipurpose system is needed to be developed which consists of different spline methods and uses the particular spline that is most suited for the desired type of shape control. Thus, to avoid a multiplicity of methods, one method, with holistic approach, is presented which can suffice and is capable of generating a broad range of spline curves, is easy to implement, provides a shape control according to the users wishes and is computationally economical.

Keywords: Splining, curve, geometric continuity.

AMS Subject Classification: 42A05, 42A10, 65D07, 65D10, 65D17.

1. INTRODUCTION

Modeling of objects is a significant area of study and practice in the world of computing today. In addition to its critical importance in the traditional fields of automobile, aircraft manufacturing, shipbuilding, shoe industry, and general product design, more recently, the re-engineering and designing methods have also proven to be indispensable in a variety of modern industries, including robotics, medical imaging, visualization, Textile, Fashion, Painting, Art, Archeology and even media and many others.

This presentation aims to provide and enlighten on the tool of splining for Modeling of objects. Specific concentration would be made on modeling by detecting geometrical features. The talk is going to focus on interdisciplinary methods and affiliate research in the area. It aims to provide the audience with a variety of techniques, applications and examples necessary for various real life problems. The major goal of the talk is to stimulate views and provide a source where researchers and practitioners can find the latest developments in the field.

The talk may specifically be of interest to people in the industries or academic fields including Computer Graphics, Computer Aided Geometric Design, Computer Vision, Image Processing, Virtual Reality, Information Visualization, Body Simulation, Engineering Disciplines, Mathematical Sciences, Font Industry, Art and Design, Film industry, Software Industry, Manufacturing Industry. It may also lead to applications like vector graphics, digitization of hand-drawn shapes, computer supported cartooning, pattern recognition, Computer Aided Design (CAD),
2. Splines

The generation of spline curves is a useful and powerful tool in CAGD. Although the splines have many elegant properties discussed in the current literature [1–3], the curves sometimes exhibit undesirable oscillations. Various methods have been developed to control the shape of a curve. Some methods are well suited for one type of shape control but not well suited for another. For this reason, a multipurpose system is needed to be developed in which consists of different spline methods and uses the particular spline that is most suited for the desired type of shape control. Thus, to avoid a multiplicity of methods, one method can suffice which is capable of generating a broad range of interpolating curves, is easy to implement, provides a shape control according to the users wishes and is computationally economical. This talk, first of all, presents a description and analysis of a cubic spline in both interpolatory as well as B-spline form. It is actually a weighted Nu spline form. Two shape parameters are introduced in its description which provide a variety of shape controls like point and interval tensions. Similarly, this talk also presents a description and analysis of a rational cubic spline in both interpolatory and B-spline form. This rational spline provides not only a computationally simple alternative to the exponential based spline under tension but also provides as $C^2$ alternative to the well-known existing $GC^2$ or $C^1$ methods [3] like cubic Nu splines of Nielson, $\beta$-spline representation of such cubics by Barsky and Beatty, $\gamma$-splines of Boehm and weighted Nu splines. This method is the generalization of the rational spline with tension. Two shape parameters are introduced in each interval which provide a variety of shape controls like biased, point and interval tensions. The talk is then extended to general piecewise rational cubics subject to a general type of continuity constraint between the pieces. We call them as Rational $\sigma$-splines. These are a generalization of most of the above mentioned methods and provide economical alternatives to the rest of them. Also the development of a local support basis for the B-Spline like representation of Rational $\sigma$-splines can be used to obtain various existing methods in the literature [1–3]. The B-Spline like basis form of the curves can also be used to solve the interpolation problems.

3. Shape preserving curves

Visualization of shaped data is one of the important problems to be solved by splines. Shape preserving problems are also part of this presentation. The spline curves here explore the shape control parameters, dependent on the derivative data, in such a way that the spline curves preserve the positive, monotonic and/or convex shapes of the data.

4. Functional approximation

This presentation is also devoted to the idea of approximation of curves [1, 2] when they are resulted through complex functions or complex data. Some methods [3] are presented as a solution to the problem. One scheme is based upon a deterministic approach using splines. The other scheme uses genetic algorithm in its formulation where the spline can have any order. These schemes automatically compute data points to minimize errors.
5. Vectorizing planar shapes

As an application of spline curves, this presentation is meant for vectorizing the planar images [3]. The idea of linear or polygonal approximation needed in various applications, including shape recognition, point-based motion estimation, coding methods, etc., in the areas of computer graphics, imaging and vision. Some important aspects related to capturing with spline approximation have been addressed. A detailed survey of many methods, in the current literature, has been made. Some commonly referred algorithms have been explained and their results are demonstrated and compared.

6. Reverse engineering

Computer-Aided Reverse Engineering (CARE) is an important area of study in the modern age of computers today. Plenty of solutions, in the advanced and modern industry are being provided as far as designing and manufacturing are concerned [1–3]. In the modern designing, scanned digital data leads to adopt contour styling which helps to guide visual acceptance after adopting some curve or surface approximation scheme. Various objects including manufactured parts or human body parts are designed and re-designed with complex free-form geometry. This trend is quite popular and can be found in various applications like vehicle body design in recent years. The wide acceptance of free-form curves and surfaces for component design can also be attributed to the advances in curve and surface modelling and their implementations in CAD/CAM/CAE/CARE systems. Splines have been utilized as underlying approximation schemes. The optimized models have been fitted over the contour data of the planar shapes for the ultimate and automatic output. The output results are visually pleasing with respect to the threshold provided by the user.

7. Conclusion

Spline curves and surfaces are utilized for the modeling of objects in various application fields. In addition to presenting a spline approach with holistic view, its applications are studies and presented in the areas of Computer Graphics, Computer Aided Geometric Design, Computer Vision, Image Processing, Virtual Reality, Information Visualization. Overall results are visually pleasing and computationally efficient.

References

MULTIDIMENSIONAL ANALOGS
OF GELFAND-LEVITAN-KREIN EQUATIONS

S. KABANIKHIN\textsuperscript{1,2,3}, M. SHISHLENIN\textsuperscript{1,2,3}, N. NOVIKOV\textsuperscript{1,3}

\textsuperscript{1}Institute of Computational Mathematics and Mathematical Geophysics, Novosibirsk, Russia
\textsuperscript{2}Sobolev Institute of Mathematics, Novosibirsk, Russia
\textsuperscript{3}Novosibirsk State University, Russia
e-mail: kabanikhin@sscc.ru, mshishlenin@ngs.ru, novikov-1989@yandex.ru

1. Introduction

We consider the method of regularization of 2D inverse coefficient problems based on the projection method and the approach of I. M. Gelfand, B. M. Levitan and M. G. Krein.


One of the advantages of our approach (for 1D inverse coefficient problems see also [3,21,22]) is that it allows one to avoid multiple solution of 2D direct problem (see also the boundary control method [1] and the globally convergent method [2]). In [10] we proved that the boundary control method and the Krein method are equivalent in 1D discrete case.

2. 2D analogy of Gelfand-Levitan equation

Let us consider the sequence of direct problems \((k = 0, \pm1, \pm2, \ldots)\)

\[
u^{(k)}_{tt} = \nu^{(k)}_{xx} + \nu^{(k)}_{yy} - q(x,y)\nu^{(k)}, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}, \quad t > 0;
\]

\[
u^{(k)}|_{t=0} = 0, \quad \nu^{(k)}_{t}|_{t=0} = \delta(x)e^{iky},
\]

\[
u^{(k)}|_{y=\pi} = \nu^{(k)}|_{y=-\pi}.
\]

**Inverse problem 1**: find function \(q(x,y)\) by known additional information

\[
u^{(k)}|_{x=0} = f^{(k)}(y,t), \quad \nu^{(k)}_{x}|_{x=0} = 0, \quad k = 0, \pm1, \pm2, \ldots
\]

The uniqueness of the inverse problem 1 can be proved using the technique in [20,23].

The inverse problem 1 can be reduced to the system of integral equations \((k = 0, \pm1, \pm2, \ldots)\) [7,11]:

\[	ilde{w}^{(k)}(x,y,t) + \int_{-\pi}^{\pi} \sum_{m} f^{(k)}_{m}'(t-s)\tilde{w}^{(m)}(x,y,s)\mathrm{d}s = -\frac{1}{2} \left[ f^{(k)}_{m}'(y,t-x) + f^{(k)}_{m}'(y,t+x) \right].
\]

(1)

Here \(|t| < x, y \in \mathbb{R}\). The system (1) is 2D analogy of the Gelfand-Levitan equation.

Note that \(q(x,y)\) can be calculated as follows

\[
q(x,y) = 4 \frac{d}{dx} \tilde{w}^{(0)}(x,y,x-0).
\]

(2)
For finding inverse problem solution $\rho(x,y)$ in point $x_0 > 0$ we have to solve the system (1) with $x = x_0$ and calculate $q(x_0,y)$ by formula (2). For numerical calculations we use $N$-approximation [6,12,14–16] of Gelfand-Levitan equation [11] e.g. we cut the system (1) putting $w^k(x,t) \equiv 0$ for all $N < |k|$ [13].

Discrete analogies of the Gelfand–Levitan equation were considered in [5,8,9,19].

3. Reconstruction of the velocity $c(x,y)$. 2D Krein equation

Inverse problem 2: find the velocity $c(x,y)$ from the sequence of relations $(k = 0, \pm 1, \pm 2, \ldots)$:

$$c^{-2}(x,y)u^{(k)}_{tt} = u^{(k)}_{xx} + u^{(k)}_{yy}, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}, \quad t > 0;$$

$$u^{(k)}|_{t=0} = 0, \quad u^{(k)}_{t}|_{t=0} = e^{i ky} \delta(x).$$

Let $\tau(x,y)$ be a solution of Cauchy problem for the eikonal equation

$$\tau_x^2 + \tau_y^2 = c^{-2}(x,y), \quad x > 0, \quad y \in \mathbb{R};$$

$$\tau|_{x=0} = 0, \quad \tau_x|_{x=0} = c^{-1}(0,y), \quad y \in \mathbb{R}. \quad (3)$$

Let us introduce new variables $z = \tau(x,y)$, $y = y$ and new functions

$$v^{(k)}(z,y,t) = u^{(k)}(x,y,t), \quad b(z,y) = c(x,y). \quad (5)$$

Since the velocity is supposed to be strictly positive this change of variables is not degenerate at least in some interval $x \in (0,h)$.

Let us consider the sequence of the auxiliary problems $(m = 0, \pm 1, \pm 2, \ldots)$ [7,11]:

$$u^{(m)}_{tt} = u^{(m)}_{xx} + b^2 u^{(m)}_{yy} + qu^{(m)}_{yy} + pu^{(m)}_{zz}, \quad z > 0, \quad y \in \mathbb{R}, \quad t \in \mathbb{R}; \quad (6)$$

$$w^{(m)}(0,y,t) = e^{imy} \theta(t), \quad w^{(m)}_z(0,y,t) = 0. \quad (7)$$

Here

$$q(z,y) = 2b^2 r_y, \quad p(z,y) = b^2 (z,y) (\tau_{xx} + \tau_{zz}). \quad (8)$$

We suppose that $c(0,y) = b(0,y)$ is known and for simplicity $b(0,y) \equiv 1$ for $y \in \mathbb{R}$.

In the neighborhood of the plane $t = z$ the solution of the direct problem (6), (7) has the form [7,11]:

$$w^{(m)}(z,y,t) = S^{(m)}(t,y) \delta (z-t) + Q^{(m)}(t,y) \theta (z-t) + w^{(m)}_1(z,y,t). \quad (9)$$

Here $w^{(m)}_1$ is continuous function and functions $S^{(m)}$ and $Q^{(m)}$ solve the following problems:

$$2S^{(m)}_t + b S^{(m)}_y + p S^{(m)} = 0, \quad t > 0, \quad y \in \mathbb{R}; \quad (10)$$

$$S^{(m)}|_{t=0} = \frac{1}{2} e^{imy}. \quad (11)$$

$$2Q^{(m)}_tt = S^{(m)}_t - \left[ b \phi_y^{(m)} + b^2 S^{(m)}_yy + p Q^{(m)} \right], \quad t > 0, \quad y \in \mathbb{R}; \quad (12)$$

$$Q^{(m)}|_{t=0} = 0. \quad (13)$$

The 2D analogy of M.G. Krein equation follows from (9) $(m = 0, \pm 1, \pm 2, \ldots)$:

$$\sum_m S^{(m)}(z,y) f^{(k)}_m(t-z) + \tilde{w}^{(k)}(z,y,t) + \sum_m \int_{-z}^{z} f^{(k)}_m(t-s) \tilde{w}^{(m)}(z,y,s)ds = 0, \quad |t| < z. \quad (14)$$

To find the solution $c(x,y)$ of the inverse problem 2 we solve the system (10)–(14).
ACKNOWLEDGEMENTS

The work was supported by RFBR (grants 16-29-15120, 18-31-00409).

Keywords: Coefficient inverse problems, Gelfand-Levitan equation, Krein equation, regularization.

AMS Subject Classification: 49N45, 65M32, 65N21.

REFERENCES

THE APPLICATION OF FUZZY LOGIC AND MULTI-FRACTAL ANALYSIS FOR RESERVOIR MANAGEMENT

B.A. SULEIMANOV, F.S. ISMAILOV, N.I. HUSEYNOVA, E.F. VELIEV

Oil Gas Scientific Research Project Institute, SOCAR, Baku, Azerbaijan
e-mail: nahide.huseynova@socar.az

1. INTRODUCTION

Management of oil fields is associated with decision-making in conditions of insufficient information. Expanding the arsenal of methods for analysis of field development indicators, increases the validity of decisions. For the system analysis of oil and gas fields development, which is an evolutionary humanistic system, both static and dynamic methods are used. Static methods are used in the study of quantitative changes without regard to qualitative changes. Dynamic analysis is applied when changes in time become the object under study [5].

The paper proposes new methods for dynamic analysis of reservoir stage development with the following features:
- to distinguish the boundaries of the reservoir development stages;
- oil production forecast for secondary and tertiary recovery methods;
- to assess the degree of self;
- organization of the development process;
- the reasonable choice of recovery methods that consistent with the stages of reservoir development.

Methods of fuzzy logic and fractal dimension theories have been used.

2. THE LIFE CYCLE OF AN OIL FIELD FROM THE STANDPOINT OF FUZZY LOGIC THEORY

The concept of the field development stages and criteria of it has been based on empirical estimates, known since the mid-70s of the last century [1]. This approach does not take into account that the stages, differing by the type of distribution function for the current oil production, do not have clearly defined time limits.

To determine the boundary points of adjacent stages within the accuracy of the oil production curve \( Q = q(t) \), we propose a new approach to the analysis of there reservoir life cycle \( T = [t_1, t_2] \), where \( t_1 = \min_{k=1, k_0} t_{ik} \), \( t_2 = \max_{k=1, k_0} t_{ik} \). Splitting \( T \) into successive intervals (stages) according to the dynamics of oil production, we consider that the intersection of adjacent intervals \( T_j \) and \( T_{j+1} \) satisfies the condition:

\[
t_{j+1}^{n_j} = t_{j+1}^n, \quad t_{j+1}^k = t_{j+1}^{k_j}. \quad (j = 1, 2, 3, 4), t_1^n = t_1, t_4^n = t_2.
\]

To identify the distribution function of oil production under \( T_j \) used a training sample \( X_j \) composed of many values, corresponding to \( t \in T_j \). Obvious that, \( T = \bigcup_{j=1}^{4} T_j \)-is a discrete subset of \( T \). Discrete set of points \( \{t, q_j\}_{t \in T_j} \) describes the dynamics of oil production changes at time interval \( \hat{O}_j \). For receiving \( \{t, q_i\}_{t \in T_j} \) Savitsky-Golay filter has been applied [3]. Smoothed values of
oil production $q_i, i=1,...,N$ correspond to the values $t_i$ defined as the left ends of the source viewport. Further, a post-stage structuring is carried out, which includes an assessment of the distribution function and an assessment of the adequacy of the chosen hypothetical distribution function $F_0(x)$ of the true distribution function $F(x)$. The method is used to determine the boundaries of adjacent stages, estimate the residual recoverable reserves and the level of current oil production.

3. **Multifractal analysis of the state of development of deposits**

Impact on the formation is accompanied by processes of self-organization. If the reservoir system has properties of inhomogeneous multifractal object, the method of multifractal fluctuation analysis (MFFA) is suitable for studying the time series of the current oil production values [4]. The method is based on the calculation of generalized fractal dimensions (Renyi dimensions) $D_q$ using generalized Hurst exponents $h(q)$ and deformation index $q > 1$ statistical partial sums corresponding to the studied time series. The recovery methods (injection of water, gas and other agents) determine the choice of the deformation parameter $q$. The dynamics of Renyi dimension time series changes for current oil production values is used as a relationship characteristic between the strain indicator $q$ and the state of the object under study. Early projections are based on the obtained results.

4. **Fractal analysis of the advancing front of water**

To control the front of the oil-water contact in the reservoir has been proposed to analyze the dynamics of the front line monofractal dimension and multi-fractal dimension of the time series of the minimum distances between the displacement front and the production wells. The line of water-oil contact in the reservoir, presented as a geometric object, is characterized by monofractal dimension. According to the theory of fractal dimension, the position of the line in the formation depends on its past positions (global determinism). Locally, the position of the front is random, because in each case are several equiprobable options of events succession. To calculate the monofractal dimension (Minkowski) $d$:

$$d = - \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(\varepsilon)}.$$

For the lines of the front oil displacement, languages, and breakthrough methods of the theory of functions of a complex potential and analytic geometry pre-determined:

- displacement front position at a fixed point in time [2]:

$$F_i(z) = F_i^1(z) + iF_i^2(z),$$

- reference coordinates of the points through which the line passes,
- the length of the line $N(\varepsilon)$ at different grid sizes $\varepsilon$ superimposed on the field map.

where:

- $F_i(z)$ is complex potential;
- $F_i^1(z)$ is the velocity potential function;
- $F_i^2(z)$ is flow function;
- $i$ is imaginary unit;
- $t(z)$ is current at time $t$ at the point with coordinate $z$. 


5. Numerical implementation of the method

The proposed algorithm has been proved on 20 fields from different regions. For example: "Guneshli" (Azerbaijan), "Uzen" (Kazakhstan), Forties (North Sea). Analysis of the field data showed that field development without considering the dynamics of reservoir stage development and propagation of the water/oil displacement front leads to oil production decline and water cut increase.

Keywords: Dynamic analysis, fractal dimension, fuzzy logic, distribution function, velocity potential, current function

AMS Subject Classification: 28A80, 03B52, 94D05, 81Q35.

References

APPLICATION OF METHODS OF DATA MINING IN THE EDUCATIONAL PROCESS

ALI ABBASOV¹, TARANA ALIYEVA²

¹Azerbaijan National Academy Sciences, Baku, Azerbaijan
²Department of Information Economy and Technologies, Azerbaijan State University of Economics, Baku, Azerbaijan
e-mail: tarana.aliyeva@unec.edu.az

1. INTRODUCTION

As information technologies become more widespread, the volumes of information stored in databases increase. Traditional approaches are becoming ineffective, and that leads to the development of modern methods of Intelligent Data Analysis (IDA), as these methods are promising directions for increasing the efficiency of analysis of large volumes of poorly structured information. The goal in this paper is to show the advantage of methods of data analysis with the help of economic and physical models of analogies.

Recently, in modern organizations, continuous development requires an increasingly effective and efficient organizational and production environment and the availability of tools for modeling business processes, analyzing their implementation, monitoring and documentation tools. Modeling a business process is a complex task that requires a thorough knowledge of the process. In solving this type of problem, the analyst is provided with means for testing hypotheses in the analysis of data. Further, the analyst generates hypotheses based on his knowledge and experience. However, knowledge is not only in man, but also in the accumulated data that are analyzed. Such knowledge is often called “hidden”, as they are contained in gigabytes and terabytes of information that a person is not able to investigate on his own. That is why in modern science there is an increasing interest in discovering hidden knowledge mostly by means of IA, which is given special attention in this work. The purpose of this technology is to study the process of finding new, valid and potentially useful knowledge in the database and finding them necessary for making optimal decisions in various areas of human activity - in science, business and telecommunications, banking, industrial production and so on.

From various aspects of the problem of analyzing the data of the educational process were considered in the works of Baker R. [1], Grigoriev L.I. [2]. In this paper, the methods and technologies of experimental and theoretical studies of the regularities of the development of the system of higher professional education are studied, and an approach is proposed for quality management in the education process with the use of IDA tools.

2. THE APPLICATION OF THE METHOD OF DATA ANALYSIS IN THE FIELD OF EDUCATION

Data Analysis plays an important role regardless of the type of industry, and in the sense of the education system. Conducting effective policy and support for the adoption of reforms in the field of education requires the use of new methods of analysis to prepare organizational and management solutions that are adequate to modern tasks. In this situation, information and analytical support becomes one of the main "services" in solving the problem of modernizing the
quality management of education. Education - one of the key links in the socio-economic system. In the conditions of formation of the economy based on knowledge, educational institutions of higher professional education should play a major role. But their missions, structures and management systems need to change in order to meet the current requirements of the economy and the future requirements of a knowledge-based economy. In accordance with the former (before the 90-ies) norms, the education system was supposed to prepare specialists for orders, centrally formed by the state. Under the conditions of the fundamentally new structure of the country’s economy, the practical disappearance of individual enterprises and entire industries, the growth of medium and small businesses, this system ceased to exist.

The question now needs to be put as follows. What should be done in conditions when some of the graduates do not seek or can't work in their specialty, and some in general plan or are forced to leave the country; most employers do not want or can't train them; some professors can't provide the knowledge that is in demand now, since they have never worked in modern organizations and modern industries, or simply they do not have the time and desire for it? Is it not worth the question now that the previously existing system of higher professional education should radically change, because, at present, different categories of participants in the educational process and the education system as a whole have different goals and interests.

Building an IA model is an integral part of a larger process, starting with the definition of the basic problem that the model will solve, and ending with the deployment of this model in the production environment. Despite the fact that this process is cyclical, each step does not necessarily lead directly to the next step. With insufficient data, additional data are needed to create the required IA models. At present, the credit technology of education is widely used in Azerbaijan's universities. Information systems of Azerbaijani universities accumulate large volumes of information about students' learning activities, students' academic achievements are fixed during the semester with the help of control points, and the final grade is calculated. Existing information systems in higher education institutions (if they exist) are used only as information support systems for the educational process and the accounting system, but not as a management system. They lack the components necessary for analysis, modeling and forecasting the behavior of the elements of the university system and the university system as a whole. This does not allow real control over processes, resources and, ultimately, the educational system.

It is necessary to use intelligent algorithms for processing information that could provide clear and understandable results for making decisions in order to improve the learning process. However, the problems of applying the methods of "Data Mining" for the analysis of data and decision-making in the sphere of education remain unresolved, taking into account various criteria. The purpose of the study is to analyze the learning outcomes and activities of students to make management and organizational decisions using the methodology of operational and IDA.

To control the quality of the educational process, the authors proposed an approach that includes methods of OLAP technologies and "Data Mining", which allows:

- identify the patterns and trends that exist in education data systems;
- form clusters containing objects of education with similar characteristics;
- find dependencies in large data sets;
- identify the indicators of education that best allow you to predict the results of the educational process;
- build models for predicting the results of educational activities;
- identify the weak and strong points of educational policy;
- generate recommendations for making managerial decisions [3].

To implement the proposed approach, primary data were processed and analyzed with numerous indicators of the educational statistics of a higher education institution for a certain period of time. The algorithm for implementing the proposed approach in the following order:

1. Collection and cleaning of the educational process data.
2. The study of data, which will allow us to understand how adequately the prepared set represents the educational process of the university.

3. Select the kind of analysis: OLAP analysis or Data Mining.

   a) OLAP-based solution allows implementing fast aggregation / detailed data operations on an arbitrary set of indicators, thus providing the analyst with detailed or generalized operational information on the indicators of the educational process that interest him. For our analysis, as measurements in which the data will be analyzed, the following can act:
   - indicators of the educational process (grades for the exam, the final score, colloquiums 1 and 2, etc.);
   - period (depending on the degree of detailing year, semester, rating period, week);
   - faculty;
   - aggregation level (department, specialty, group). A multidimensional model is visually represented using a cube.

   b) Intelligent analysis, which consists of the following stages: analysis of the influence of factors, factor analysis, cluster analysis.

   The proposed approach is implemented in a Microsoft Power BI environment using an integrated Excel environment to create and operate this mining model. An analysis of the influence of factors makes it possible to determine how the result of the educational process depends on other parameters (learning factors). In this study, an analysis was made of the influence of different parameters of the educational process on each other, while completely independent and, conversely, completely dependent factors should be removed from consideration. Thus, the analysis of the problem shows that the task of managing the quality of education in modern conditions is complex and diverse requires the simultaneous use of several approaches to management, accounting for many factors. The existing approach to the management of the quality of education in higher education has a number of such drawbacks as the low level of analytical processing of the educational process data, therefore, a new approach to the management of the quality of the educational system using the methods of operational and intellectual data analysis was proposed, and the architecture of the information and analytical system for management was designed quality of the educational process. The results of operational and intellectual analysis make it possible to improve managerial activity in the sphere of education and can be used in decision support systems.

**Keywords:** Intelligent data analysis, OLAP, factor analysis, cluster analysis.

**AMS Subject Classification:** 62-07, 68U35.

**References**


SECOND-ORDER COEXHAUSTERS CALCULUS

M.E. ABBASOV

1St. Petersburg State University, SPbSU,
7/9 Universitetskaya nab., St. Petersburg, 199034 Russia
e-mail: abbasov.majid@gmail.com, m.abbasov@spbu.ru

Abstract. Coexhasuter is a new notion in nonsmooth analysis that allows one to study extremal properties of a wide class of functions [1–3]. This class is introduced in a constructive manner analogous to the “classical” smooth case. Formulas of calculus were developed. Coexhausters are families of convex compact sets allowing one to represent the main part of the increment of the studied function in the form of MaxMin or MiniMax of affine functions.

For a more detailed study of nonsmooth functions a notion of second-order coexhausters was introduced. These are also families of convex compact sets which are used to represent the main part of the increment of the studied function in the form of a MaxMin or MiniMax of quadratic functions. These objects are used to build second-order optimization algorithms. However, an important problem of constructing calculus arises again. The solution of this problem is the subject of this work.

Keywords: Nonsmooth analysis, nondifferentiable optimization, second order coexhausters.

AMS Subject Classification: 49J52, 90C47.

We will say that a function $f$ has a second-order upper coexhauster in the sense of Dini at a point $x$ if the following representation is valid:

$$f(x + \Delta) = f(x) + \min_{C \in \mathcal{E}(x)} \max_{[a,v,A] \in C} \left[ a + \langle v, \Delta \rangle + \frac{1}{2} \langle A \Delta, \Delta \rangle \right] + o_x(\Delta^2),$$

(1)

where

$$\lim_{\alpha \downarrow 0} \frac{o_x((\alpha \Delta)^2)}{\alpha^2} = 0 \quad \forall \Delta \in \mathbb{R}^n,$$

$\mathcal{E}(x)$ is a family of convex compact sets in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n}$. If

$$\lim_{||\Delta|| \to 0} \frac{o_x(||\Delta||^2)}{||\Delta||^2} = 0,$$

in (1) then the function $f$ is said to have a second-order upper coexhauster in the sense of Hadamard at the point $x$.

The set $\mathcal{E}(x)$ is called a second-order upper coexhauster (in the sense of Dini or Hadamard) of the function $f$ at the point $x$.

Similarly, we say that a function $f$ has a second-order lower coexhauster in the sense of Dini at a point $x$ if the following representation is valid:

$$f(x + \Delta) = f(x) + \max_{C \in \mathcal{E}(x)} \min_{[a,v,A] \in C} \left[ a + \langle v, \Delta \rangle + \frac{1}{2} \langle A \Delta, \Delta \rangle \right] + o_x(\Delta^2),$$

(2)
where \( \lim_{\alpha \to 0} \frac{o_x((\alpha \Delta)^2)}{\alpha^2} = 0 \), for all \( \Delta \in \mathbb{R}^n \), \( E(x) \) is a family of convex compact sets in \( \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \). If \( \lim_{||\Delta|| \to 0} \frac{o_x(||\Delta||^2)}{||\Delta||^2} = 0 \) in (2) then the function \( f \) is said to have a second-order lower coexhauster in the sense of Hadamard at the point \( x \).

The set \( E(x) \) is called a second-order lower coexhauster (in the sense of Dini or Hadamard) of the function \( f \) at the point \( x \).

Without loss of generality, the matrices \( A \) can be assumed to be symmetric.

**Theorem 1.** Let functions \( f_1 \) and \( f_2 \) have second-order coexhausters at the point \( x \in X \)

\[
E_1(x) = [E_1(x), E_1(x)], \quad E_2(x) = [E_2(x), E_2(x)],
\]

respectively. Then the function \( F = \lambda_1 f_1 + \lambda_2 f_2 \) have second-order coexhauster at the point \( x \)

\[
E(x) = [E(x), E(x)], \quad E(x) = \lambda_1 E_1(x) + \lambda_2 E_2(x),
\]

where

\[
\lambda E = \begin{cases} \lambda E, \lambda \geq 0, \\ \lambda E, \lambda < 0. \end{cases}
\]

We will say that the families \( E_i(x), E_i(x), i = 1, 2 \) are jointly bounded if there exists \( M > 0 \), such that for any \( C \) from \( E_i(x) \) or \( E_i(x), i = 1, 2 \) the inequality

\[
||z|| \leq M \quad \forall z = [a, v, A] \in C,
\]

holds. It is obvious that if the families \( E_i(x), E_i(x), i = 1, 2 \) are jointly bounded, there exists \( R > 0 \) such that for any \( C \) from \( E_i(x) \) and \( E_i(x), i = 1, 2 \) inequality

\[
\left| a + \langle v, \Delta \rangle + \frac{1}{2} \langle \Delta, A\Delta \rangle \right| < R \quad \forall \Delta \in B_1(0), \quad \forall [a, v, A] \in C
\]

is valid, where \( B_1(0) \) is the unit ball centered at the origin.

**Theorem 2.** Let functions \( f_1 \) and \( f_2 \) have second-order coexhausters at the point \( x \in X \)

\[
E_1(x) = [E_1(x), E_1(x)], \quad E_2(x) = [E_2(x), E_2(x)]
\]

respectively. Then the function \( F = f_1 \cdot f_2 \) also has second-order coexhauster \( E(x) = [E(x), E(x)] \) at the point \( x \) in the form

\[
E(x) = -R[ E_1(x) + E_2(x)] + \tilde{E}(x) + f_1 E_2(x) + f_2 E_1(x),
\]

where

\[
\tilde{E}(x) = \left\{ C = [a, v, A] \left| a = a_1 a_2 + R(a_1 + a_2); \quad v = (v_1 + v_2)R + a_1 v_2 + \\
+ a_2 v_1; \quad A = A_1(a_2 + R) + A_2(a_1 + R) + 2 v_1 v_2^T \right\}
\]

\[
[ a_1, v_1, A_1] \in E_1(x), \quad [ a_2, v_2, A_2] \in E_2(x),
\]

and

\[
E(x) = \left\{ C = [a, v, A] \left| a = a_1 a_2 + R(a_1 + a_2); \quad v = (v_1 + v_2)R + a_1 v_2 + \\
+ a_2 v_1; \quad A = A_1(a_2 + R) + A_2(a_1 + R) + 2 v_1 v_2^T \right\}
\]

\[
[ a_1, v_1, A_1] \in E_1(x), \quad [ a_2, v_2, A_2] \in E_2(x).
\]
Theorem 3. Let the function $f$ has second-order coexhauster at the point $x \in X$ $E(x) = [E_1(x), E_1(x)]$ and the families $E_i(x)$ and $E_i(x)$ are jointly bounded. Then the function $F(x) = \frac{1}{f_i}$ also has second-order coexhauster $E(x) = [E(x), E(x)]$ at the point $x$ in the form

$$E(x) = -\frac{E_1(x)}{f_1(x)} + \frac{1}{f_1(x)} \left( -2R(x) + 2f(x)E(x) + \tilde{E} \right),$$

$$E(x) = -\frac{E_1(x)}{f_1(x)} + \frac{1}{f_1(x)} \left( -2R(x) + 2f(x)E(x) + \tilde{E} \right),$$

$$\tilde{E}(x) = \left\{ C = [a, v, A] \mid a = a_1a_2 + R(a_1 + a_2); v = (v_1 + v_2)R + a_1v_2 + a_2v_1; A = A_1(a_2 + R) + A_2(a_1 + R) + 2v_1v_2 \right\}$$

$$[a_1, v_1, A_1] \in E_1(x), [a_2, v_2, A_2] \in E_1(x),$$

$$\tilde{E}(x) = \left\{ C = [a, v, A] \mid a = a_1a_2 + R(a_1 + a_2); v = (v_1 + v_2)R + a_1v_2 + a_2v_1; A = A_1(a_2 + R) + A_2(a_1 + R) + 2v_1v_2 \right\}$$

$$[a_1, v_1, A_1] \in E_1(x), [a_2, v_2, A_2] \in E_1(x).$$

Theorem 4. Let functions $f_i$ have second-order coexhauster at the point $x \in X$ for all $i$ from a finite set $I$, $E_i(x) = [E_i(x), E_i(x)]$. Then

1. the function $F_1 = \max_{i \in I} f_i$ has a lower second-order coexhauster at the point $x$ in the form

$$E_1(x) = \bigcup_{i \in I} E_i(x),$$

where

$$\hat{E}_i(x) = \left\{ C + [f_i(x) - F_1(x), 0, 0 \times 0] \mid C \in E_i(x) \right\},$$

2. the function $F_2 = \min_{i \in I} f_i$ has an upper second-order coexhauster at the point $x$ in the form

$$E_2(x) = \bigcup_{i \in I} \hat{E}_i(x),$$

where

$$\tilde{E}_i(x) = \left\{ C + [f_i(x) - F_2(x), 0, 0 \times 0] \mid C \in E_i(x) \right\}. $$

Acknowledgments

The reported study was partially supported by RFBR, research project No. 18-31-00014.

References


1. Introduction

In applications, theorems on continuous dependence of solutions on initial data are of great importance. They guarantee that close solutions of the equations under consideration should correspond to close initial points. By the same token, the acuteness of the problem related to impossibility of exact measurements or coordinates of initial points under identification is weakening. Since the measurement process is always performed with some accuracy. In connection with this, when identifying, the definition of a particular starting point is realized from a certain small neighborhood of the initial point given together with the equation. Thus, even in classical problems, when studying the solutions of equations, it may be necessary to consider the neighborhood instead of one initial point. Below, we propose a new interpretation of the use of the concept of a small neighborhood of the initial point, and we generalize this concept for all points of the solutions of the equations under consideration.

The concept of a material point is universal. It is usually used in describing the behavior of solids, liquids, gases and plasma. The material point is a usual mathematical point (a geometrical point in the sense of Euclid’s definition [3, p.248]) with a physical parameter (mass, charge, density, pressure, temperature, etc.) related to it. When describing the behavior of an object, it is possible to use one or more material points. For example, in the description of the translationally curvilinear motion of a solid, one selected material point is usually used. It is considered that this selected point is rigidly (immobile) connected with a solid and, quite often, can be inside this body (often combined with the center of gravity of a solid). As a result, in the case of a translationally curved motion of a solid, any other point that is rigidly connected with the solid describes in space a trajectory parallel to the trajectory of the chosen point. Therefore, in the case under consideration it turns out to be sufficient to derive the equation of motion only for the chosen material point. A similar situation arises in the derivation of the equations of motion for laminar fluid and gas flows. However, for turbulent flows the situation changes. It is impossible to assert that any two neighboring material points in such a flow will always describe parallel current lines (otherwise, the flow would not be turbulent, but laminar). In a turbulent flow, the streamlines may admit vortices, they can lose the continuity property, becoming even pointwise discontinuous. Therefore, in the general case, it is impossible to use the continuity equation to describe turbulent flows. In our previous report at this conference, we also pointed out the inadmissibility of the description of turbulent flows in the general case by the Navier-Stokes equation [4, 5].
2. Problem statement and main results

Now let’s pay attention to the following two fundamental positions, which are usually used by default in deriving the equations of motion (behavior):

1. In the derivation of the equations either the linear dimensions (overall dimensions) of the substance whose behavior is being investigated are either not taken into account, or are taken into account only linearly and incompletely.

2. The material point chosen for derivation of equations is assumed rigidly (absolutely fixed) connected only by one point of the object whose behavior is intended to be described by this point (the number of selected points is usually considered to be finite).

Let us explain the second part of the proposition. For example, an indirect and incomplete accounting of only one of the three linear dimensions (called the characteristic diameter) is carried out with the definitions of a numbers of Reynolds (Re), of Prandtl (Pr), Grashof (Gr), of Fourier (Fo), of Nusselt (Nu) and so on. These dimensionless numbers are similarity criteria. These numbers are assumed to be constant, which is justified when the equations used to describe the flow (including such constants) are invariant relative to one-parameter group of extensions with respect to parameters included in the definitions of these numbers.

Now we are ready to present our approach to these problems.

First of all, we will reject both fundamental statements described above. Rejection from the position I means that we start with the linear dimensions of the objects whose behavior is being investigated. Naturally, we assume that each object under study in 3-dimensional Euclidean space has linear dimensions, and therefore includes a certain volume. The surface of this volume can have a complex shape and differ from object to object. To give some universality to our approach, we will proceed from the fact that the volume of each object entirely contains some fixed (rigidly connected) relative to this object smaller volume of the standard form possessing a surface of simple structure. However, we will usually prefer spherical proper volumes (hereinafter briefly denoted by the $S$-volume symbol), which usually has its own fixed diameter for each individual substance (i.e. the diameters of the proper balls may differ for different objects or flows, and, if necessary, they can even be variables).

The preference that we gave to the ball in choosing its proper volume is to some extent connected with the following consideration. It can be expected that the significance of the proposed approach will increase even more if as the diameter for the intrinsic volume, we take a number that is minimal among the errors allowed in the measurement of coordinates in the identification.

In addition, the choice of the ball has a geometric overtone. More exactly, it turns out that if the notion of a point is initially defined in the form of a sphere with a fixed diameter, and the notion of a straight line in the form of a cylinder of the same diameter, then the axioms of geometry with respect to the set of such points - spheres and lines - cylinders will be observed [3, p.44-46]. That is, we get another interpretation for Euclid’s geometry.

In the case of studying the motion of solids or laminar flows, we will assume that their motions are completely determined with the corresponding $S$-volumes (since the $S$-volume selected belongs to the object and is absolutely fixed with respect to this object). In the case of studying turbulent liquids or gases, we shall assume that the diameter of the corresponding proper volume ($S$-volume) is so small that it would be possible to neglect all possible changes in the form in the flow (and, if necessary, the radius can be considered variable).

Refusal of the II position is expressed for any object in violation of the rigid connection between the material point chosen for obtaining the equation and a certain point of the same object. A rigid connection between these points is replaced by a rigid connection between the object and its proper volume ($S$-volume). Further, the trajectory of any mathematical point belonging to its proper volume ($S$-volume) will be called an esl-trajectory or an esl-solution (the word "esl" in Azerbaijani means "real", and it is also pronounced as in the English transcription [æsl]). Thus the motion $S$-volume determines an infinite set of esl-solutions.
If we speak about determination of the coordinates of an object at some point in time, then any mathematical point from $S$-volume is suitable for this role, since this point belongs to $S$-volume, and $S$-volume completely immovably enters the structure of the object. In this case, there is no need for a rigid connection between an arbitrarily chosen mathematical point and $S$-volume. That is, in the course of time, the selected mathematical point can make any movement within the $S$-volume. As a result, esl-solutions can be continuous or not continuous, in particular, they can even be pointwise discontinuous. Therefore, esl-solutions cover any types of flow behavior that are only possible in turbulent flows.

If we are talking about the determination of the dynamic parameters of the physical indicators of a substance at a certain instant of time, then in a small neighborhood of this time we can try to determine these indicators along the trajectory of any interior point that is fixed with respect to $S$-volume.

For visibility of presentation, we consider the case of one-dimensional motion. Obviously, in this case the proper sphere turns into a segment with a length equal to the diameter of this sphere. Let $l$ be the length of this segment. We consider the Cauchy problem for an one-dimensional ordinary differential equation of the following form

$$\frac{dx(t)}{dt} = f(x(t), t), \quad t \in T = [t_0, t_1], \quad f \in C(T), \quad x(t_0) = x_0. \tag{1}$$

For the Cauchy problem (1), (2), the esl-solutions can be determined from the relation:

$$z(t) = x(t) + l \cdot \mu(t), \quad |\mu(t)| \leq 1, \quad t \in T. \tag{3}$$

If we restrict our attention to smooth esl-solutions, then using (3) and assuming that

$$\mu \in M = \{\mu = \mu(t)|\mu(t)| \leq 1, \quad \mu \in C^1(T), \quad t \in T\},$$

we obtain the following equation and the initial condition for such solutions:

$$\frac{dz(t)}{dt} = f(z(t) - l \cdot \mu(t), t) + l \cdot \frac{d\mu(t)}{dt}, \quad t \in T; \tag{4}$$

$$z(t_0) = x_0 + l \cdot \mu(t_0). \tag{5}$$

The problem (4), (5) for any function $\mu \in M$ determines a unique smooth esl-solution. Moreover, the family (4), (5) defines the set of all possible smooth esl-solutions. Further, equations of the type (4) will sometimes be called esl-equations. We note that a number of results on the theory of esl-equations are given in [1, 2]. In equation (4), considering the functions $\mu \in M$ as controls, we can formulate and investigate a number of optimization problems. With the account of the announced opportunities for the study of turbulent motions, we can hope that there are great opportunities for the further development of the theory of esl-equations.

**Keywords:** Turbulent flow, esl-solution, esl-equation, $S$-volume.

**AMS Subject Classification:** 93A30, 97M10.

**References**


SOME ISSUES OF APPLICATION OF INTERNET OF THINGS IN THE OIL AND GAS COMPLEX

RASIM ALGULIYEV¹, TAHAMSI FATALIYEV¹, SHAKIR MEHDIYEV¹

¹Institute of Information Technology, Azerbaijan National Academy of Sciences, Baku
e-mail: secretary@iit.science.az, depart3@iit.science.az, depart11@iit.science.az

1. INTRODUCTION

Locating modernized solutions is an urgent task serving to increase the productivity and competitiveness of the oil and gas complex (OGC). The use of modern information technologies in this direction is constantly developing, which provides an increase in the speed of exploration and detection of oil, an increase in oil production and a reduction in risks to health, human security, and the environment. The Internet of Things (IoT) in the OGC, as in all industrial sectors, has great prospects from an economic point of view [1, 3]. However, the use of this technology makes it necessary to solve a number of scientific, theoretical and technological problems. The article is dedicated to study of these issues and the development of a conceptual model for the use of IoT in the Azerbaijani oil company SOCAR.

The main technical factors of the formation and development of IoT include the following:

- Evolution of mini, micro, and nano-sensor production technologies with the ability to collect various information (temperature, pressure, vibration, distance, position, angle of rotation, chemical composition of the substance, etc.) from the control and management facilities;
- Transition to IPv6 technology, which removes restrictions on the number of sensors and devices that are connected;
- Introduction of wireless communication technologies that enable to directly extract information from sensors installed in the real measurement zone of parameters of various technological processes;
- Development and improvement of cloud, fog, and dew structures that help to store large volumes of information and enable the application of complex analytical tools such as Big Data, Data Mining, OLAP, Pattern Recognition, etc.

The key element of IoT is the sensor network topology. Sensor networks consist of local nodes. Each node is equipped with a sensor for data acquisition, a microprocessor for initial processing of data and development of control actions on actuators and a transceiver for receiving or transmitting data to the next node in the hierarchy [2]. As a rule, nodes of sensor networks operate in continuous mode or in a mode on demand. In the first mode, the network node uninterruptedly receives the data and sends it online or after the primary processing to the neighboring or central node. In the second mode, the node is in hibernation mode, waiting for the command from the neighboring or central node. Wireless devices are traditionally connected through the radio frequency spectrum. RFID, Bluetooth, Wi-Fi, ZigBee are usually used at the level of short-range nodes (hundreds of meters within one field), while cellular or satellite communication is used for long-range wireless communications (offshore platforms, main pipeline monitoring systems). Unambiguously, the transition to Internet Protocol version 6 (IPv6) will
allow to have a unique IP address for each sensor, node or device. However, there are problems associated with several aspects of implementing IPv6, which include security management, the implementation of interfaces supporting the dual IPv6 and IPv4 environment and the adoption of new standards. In recent years, various solutions have been developed based on the SCADA, M2M, and WSN dispatching and data collection systems. In [4], examples are given of devices for monitoring the state of equipment, in the production of petroleum products, monitoring of pipelines, cathodic protection stations for pipelines, detection of corrosion, wellhead monitoring, pumping installations, the system when drilling oil wells. So it is suggested to use sensors distributed along the entire length of the pipeline at fixed points for in-pipe inspection, but it is not possible to perform a measurement very close to a leak. In the technology of IoT, the physical parameters measured by the sensors can become the basis for predicting the maintenance of the equipment according to its actual state.

2. **Description of technological processes in OGC**

It is necessary to pass several stages of technological processes in the OGC before the final products derived from oil and natural gas will be offered to the consumer. At the first stage, a search for potential hydrocarbon fields (oil and natural gas), exploratory drilling and other works is performed. The second stage consists of the extraction of raw materials, that is, the extraction of oil or natural gas from the earth’s interior from offshore platforms or on land. At the third stage, raw materials are transported and delivered to consumers for further processing. For example, extracted oil with impurities passes through pipelines and is pumped into primary battery tanks, where oil is separated from gas, and water. The crude oil is then stored in storage tanks, from where it travels through oil trunk pipelines to oil refineries, to other storage tanks, tanker vessels or tank wagons for transportation. Pumping stations are installed at regular intervals along the entire length of the route to pump oil through the pipelines. Pumps are used to initiate and maintain pressure, overcome friction, account for the difference in altitude along the length of the route, and other factors. In the fourth stage, oil or natural gas is processed to produce final products such as gasoline, kerosene, jet fuel, diesel fuel, fuel oil, lubricating oils, liquefied gas, plastics, and other materials. These technological processes occur in the three main sectors of the OGC or as it is customarily called, Upstream, Midstream, and Downstream.

3. **Proposed conceptual model on IoT basis**

Since the emergence and development of microprocessors and network devices, the possibility of using microcontrollers, supplemented by sensors and mechanisms, has been actively studied to ensure greater reliability, efficiency, and security of production processes in the OGC (geological exploration, drilling, extraction, processing, transportation, etc.), as there is a high level of financial, environmental, and humanitarian risks. Traditionally, information flows processing and management in oil and gas producing enterprises occur on three levels. At the lower level, data monitoring, data collection from sensors and primary processing of information for the purpose of developing control actions on oil and gas production facilities is carried out in real time with the help of local-group devices. Replacement of conservative and mostly manual control and monitoring devices and the provision of production, transportation and processing processes in the OGC with new, easy-to-install sensors allows for continuous automatic control of technological processes, registration and storage of data, and remote configuration. Thus, it is possible to increase reliability, security, energy efficiency, and influence on environmental indicators, such as gas emissions, leaks and spills of primary raw materials. At the next level, decisions are made on optimizing processes, determining the frequency of repair activities to reduce downtime and optimizing maintenance intervals for units and assemblies, ensuring efficient operation, etc. Unplanned downtime due to equipment breakdowns that lead to loss of time and finances can be reduced through introduction of intelligent maintenance systems. The upper level is the level of the company on which the analysis (big data processing) is implemented,
which results in the coordination of activities that are part of the company of enterprises and structures to achieve overall efficiency, measures are taken to increase security and reduce risks. A concept of IoT [5] determines the development of industry in the coming years. A prerequisite for the operation of any production facility, including the OGC within the framework of this concept, is the direct information interaction of various types of facilities equipped with various sensors, the availability of intelligent devices that can transmit data, make decisions and interact with each other. A concept model based on IoT can be presented at the following levels:

- Level one. Control object with built-in sensors.
- Level two. Gateways controlling data flows. They can also perform primary processing and release of control actions for level one.
- Level three. Clusters of real-time data processing.
- Level four. Cloud infrastructure, which includes a processing center and a database.

The main stages of technological processes are spatially-distributed. They are grouped into clusters according to certain features; data is processed in clusters without the need to transfer to the cloud. Thus, there is a redistribution of the load from the cloud service to fog computing. To increase the reliability and efficiency of management in case of failures or channel congestion, virtual cross-links are created between the corresponding nodes: sensor-sensor, gateway-gateway, gateway-fog, and fog-fog.

4. Conclusion

Currently, the OGC faces new production problems, especially against the background of a decline in oil prices. Finding new ways to increase efficiency and competitiveness, improve results, and reduce costs is an urgent and important task. Here a special role is assigned to the control and collection of detailed and accurate data and information on the production process. The use of IoT in these processes is the most optimal strategy. IoT has the potential capabilities to manage the main processes for the three sectors of the OGC with more efficient and reliable results. The processing of large data collected using new technologies can be performed using the capabilities of cloud technologies, Big Data, and data mining technology, and the obtained results will provide operational and look-ahead control, thereby increasing production efficiency.

5. Acknowledgment

This work was supported by the Science Development Foundation under the SOCAR-Grant No.01LR-ANAS.

Keywords: Internet of things, oil and gas complex, M2M, WSN, SCADA.

AMS Subject Classification: 68M11.

References

CONVERGENCE AND STABILITY OF NEW ITERATIVE SCHEME IN BANACH SPACE AND APPLICATION

JAVID ALI¹, FAEEM ALI²

¹Department of Mathematics, Aligarh Muslim University, India
e-mail: javid.mm@amu.ac.in

Abstract. In this talk, we introduce a new iterative scheme for contraction mapping in real Banach space. We also prove some convergence and stability results for this new proposed scheme. We show that our scheme converge faster than several well known schemes. For numerical result, we use MATLAB. Further, we apply proposed scheme to approximate solution of integral equation.

Keywords: Contraction mapping, fixed points, iterative scheme, Banach space, Opial’s condition.

AMS Subject Classification: 47H09, 47H10.

1. Introduction

Throughout this paper, N denotes the set of all positive integers. We consider that C is nonempty subset of a Banach space X and F(T), the set of all fixed points of the mapping T on C.

A mapping T : C → C is said to be non-expansive if ∥Tx - Ty∥ ≤ ∥x - y∥, for all x, y ∈ X. It is called quasi non-expansive if F(T) ≠ ∅ and ∥Tx - p∥ ≤ ∥x - p∥, for all x ∈ C and p ∈ F(T). We know that F(T) is nonempty in the case when C is bounded closed convex subset of uniformly convex space X and T is non-expansive mapping, (cf. [1]).

Fixed point theory plays an important role in mathematics and it provides useful tools to solve many linear and nonlinear problems that have many applications in different fields like Engineering, Differential equation, Integral equation, Economics, Chemistry, Game theory etc. However, when the existence of fixed point of some operators is accomplished, then to find that fixed point is not an easy task, thats why we use iteration processes for computing them. A large number of researchers introduced and studied many iteration processes for computing fixed points for different mappings. In several cases, there can be more than one iteration process to computing fixed points of a particular mapping. In such cases, the speed of iterations do matter, the better speed of iterative schemes to approximate fixed point save time. The following definitions about the speed of convergence of iteration processes are due to Berinde [2].

Definition 1. Let {αₙ} and {βₙ} be two sequences of real numbers that converges to α and β respectively. Assume that

$$\ell = \lim_{n \to \infty} \frac{|\alpha_n - \alpha|}{|\beta_n - \beta|}.$$

(i) If ℓ = 0, then we say that {αₙ} converges to α faster than {βₙ} to β.
(ii) If 0 < ℓ < ∞, then {αₙ} and {βₙ} have the same rate of convergence.
Definition 2. Suppose that \( \{x_n\} \) and \( \{y_n\} \) be two fixed point iteration processes both converging to same point \( p \) of a mapping with error estimates

\[
\begin{align*}
|x_n - p| & \leq \alpha_n, \\
|y_n - p| & \leq \beta_n.
\end{align*}
\]

If \( \lim_{n \to \infty} \frac{\alpha_n}{\beta_n} = 0 \), then \( \{x_n\} \) converges faster than \( \{y_n\} \) and \( \{y_n\} \) slower than \( \{x_n\} \).

Definition 3. Let \( \{t_n\} \) be an arbitrary sequence in a subset \( C \) of Banach space \( X \). Then an iteration procedure \( x_{n+1} = f(T, x_n) \) for some function \( f \), converging to fixed point \( p \), is said to be \( T \)-stable or stable with respect to \( T \), if for \( \epsilon_n = \|t_{n+1} - f(T, t_n)\|, n \in N_0 \), we have

\[
\text{lim}_{n \to \infty} \epsilon_n = 0 \iff \lim_{n \to \infty} t_n = p.
\]

Lemma 1. Let \( \{\epsilon_n\} \) and \( \{u_n\} \) be any two sequences of positive real numbers satisfying \( u_{n+1} \leq \delta u_n + \epsilon_n, n \in N_0 \), where \( 0 \leq \delta < 1 \). If \( \lim \epsilon_n = 0 \) then \( \lim u_n = 0 \).

In Banach contraction principle, fixed points of contraction mappings can be approximated by Picard iteration [9] where the sequence \( \{x_n\} \) is generated from an arbitrary \( x_1 \in C \) as:

\[
\begin{align*}
x &= x_1 \in C, \\
x_{n+1} &= Tx_n, n \in N.
\end{align*}
\]

It is well known that, Picard iteration for a non-expansive mapping need not converge to a fixed point. Therefore, in 1953, Mann [7] introduced an iterative scheme, which has been extensively used to approximate fixed point of non-expansive mappings. In this iterative scheme, the sequence \( \{x_n\} \) is generated from an arbitrary \( x_1 \in C \), in the following manner:

\[
\begin{align*}
x &= x_1 \in C, \\
x_{n+1} &= (1 - a_n)x_n + a_nTx_n, n \in N,
\end{align*}
\]

where \( \{a_n\} \) be real sequence in \((0, 1)\), satisfying suitable conditions. It is also known that Mann iteration fail to converge to fixed points of pseudo-contractive mappings.

So, in 1974, Ishikawa [5] introduced a two step Mann iterative scheme to approximate fixed point of pseudo-contractive mappings, where the sequence \( \{x_n\} \) is defined as follows:

\[
\begin{align*}
x &= x_1 \in C, \\
x_{n+1} &= (1 - a_n)x_n + a_nTy_n, \\
y_n &= (1 - b_n)x_n + b_nTx_n, n \in N,
\end{align*}
\]

where \( \{a_n\} \) and \( \{b_n\} \) are real sequences in \((0, 1)\), satisfying appropriate conditions. Rhoades [10] made an interesting remark on the rate of convergence of these iteration processes that: Mann scheme for decreasing functions converges faster than Ishikawa scheme. For increasing functions Ishikawa iteration process is better than the Mann iteration process, also Mann iteration process appears to be independent of the initial guess (see also [11]).

In 2000, Noor [8] introduced the following iterative scheme for general variational inequalities, in this scheme, \( \{x_n\} \) is defined as:

\[
\begin{align*}
x &= x_1 \in C, \\
x_{n+1} &= (1 - a_n)x_n + a_nTy_n, \\
y_n &= (1 - b_n)x_n + b_nTz_n, \\
z_n &= (1 - c_n)x_n + c_nTx_n, n \in N,
\end{align*}
\]

where \( \{a_n\} \), \( \{b_n\} \) and \( \{c_n\} \) are real sequences in \((0, 1)\), satisfying appropriate conditions. He also studied the convergence criteria of such scheme.

condition (I) due to Senter and Dotson [12]. In 1994, Park proved weak convergence theorem for
generalized non-expansive mappings using Mann iteration in uniformly convex Banach space.

We now recall some definitions, proposition and lemmas to be used in our main results.

**Definition 4.** A Banach space $X$ is said to satisfy Opial’s property if for each weakly
convergent sequence $\{x_n\}$ in $X$ with weak limit $x$,

$$\lim_{n \to \infty} \sup_{n \to \infty} \|x_n - x\| < \lim_{n \to \infty} \sup_{n \to \infty} \|x_n - y\|$$

holds, for all $y \in X$, with $y \neq x$.

**Definition 5.** Let $C$ be a nonempty, closed and convex subset of a Banach space $X$ and let
$\{x_n\}$ be a bounded sequence in $X$. For $x \in X$, we set

$$r(x, \{x_n\}) = \lim_{n \to \infty} \sup_{n \to \infty} \|x_n - x\|$$

The asymptotic radius of $\{x_n\}$ relative to $C$ is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$$

and the asymptotic center of $\{x_n\}$ relative to $C$ is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$  

It is known that in a uniformly convex Banach space, $A(C, \{x_n\})$ consists exactly one point.

**Proposition 1.** Let $C$ be a nonempty subset of a Banach space $X$ and $T : C \to C$ be a
mapping: (i) If $T$ is non-expansive mapping then $T$ is generalized non-expansive. (ii) If $T$
is generalized non-expansive mapping and has a fixed point, then $T$ is quasi non-expansive
mapping.

**Lemma 2.** Suppose $X$ is uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all
$n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $X$ such that $\lim_{n \to \infty} \sup_{n \to \infty} \|x_n\| \leq d$, $\lim_{n \to \infty} \sup_{n \to \infty} \|y_n\| \leq d$
and $\lim_{n \to \infty} \|t_n x_n + (1 - t_n)y_n\| = d$ hold, for some $d \geq 0$. Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

**References**


TRANSFORMATION OF THE MITTAG-LEFLER FUNCTION TO AN EXponential FUNCTION AND SOME ITS APPLICATIONS TO PROBLEMS WITH A FRACTIONAL DERIVATIVE

F.A. ALIEV¹, N.A. ALIEV¹, N.A. SAFAROVA¹, K.G. GASYMOVA²

¹Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
²Azerbaijan State Pedagogical University, Baku, Azerbaijan
e-mail: f_aliev@yahoo.com, nihan.aliev@gmail.com, narchis3003@yahoo.com

Abstract. For the first time the relation between the Mittag-Lefler function and the exponential function is given. The results are applied to the construction of a solution of the Cauchy problem for ordinary linear operator differential equations with constant coefficients and fractional derivatives. The example shows that when the order of the derivatives (fractional) approaches integers then the results coincide with the classical ones.

Keywords: Exponential function, Mittag-Lefler function, fractional derivative, Cauchy problem, linear operator differential equations.

AMS Subject Classification: 44-XX.

1. Introduction

Despite the fact that recently different problems from the theory of control, optimization, gas dynamics [1, 6–8], etc. with fractional derivatives based on the Mittag-Lefler function have been developed to the present time, its relationship with exponential functions has not been established [7]. Therefore, many questions related, for example, to the theory of stability [2, 3] remain open.

In this remark we give a relation between these functions and their result is applied to finding the solution of the Cauchy problem for linear ordinary differential operator equations with constant coefficients and with fractional derivatives. Further there is given a concrete example illustrating the approximation of a solution to classical solutions, when the order of the fractional derivatives tends to integers.

2. The Mittag-Lefler Function

The Mittag-Lefler functions are represented in the form [5, 7]:

\[ \alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \alpha > 0 \]  

(1)

and

\[ \alpha,\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha > 0, \beta > 0, \]  

(2)

For further details and derivation of these functions, refer to the original sources [5, 7].
where $\Gamma(\cdot)$ is the Euler Gamma-function [7], $z$ is any real or complex variable.

Assuming the substitution $z = x^\alpha$ and, taking into account the formula $\Gamma(\alpha k + 1) = (\alpha k)!$ from (1) and (2) we obtain:

$$\alpha (x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{(\alpha k)!}, \alpha > 0,$$

and

$$\alpha, \beta (x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{(\alpha k + \beta - 1)!} = x^{1-\beta} \sum_{k=0}^{\infty} \frac{x^{\alpha k+\beta-1}}{(\alpha k + \beta - 1)!}, \alpha > 0, \beta > 0.\quad (4)$$

If we pay attention to the formula in [7]

$$D^\beta \left( \frac{x^\alpha}{\alpha!} \right) = \frac{x^{\alpha-\beta}}{\alpha - \beta}!, \beta \leq \alpha,$$

and taking into account that $\Gamma(0) = (1)! = 1$, we have further

$$\frac{x^{-1}}{(1-1)!} = \delta(x),$$

where $\delta(E)$ is the Dirac delta function [7].

3. Connection of the Mittag-Leﬄer function with exponential function $e^x$

Then, an analogue of the Euler invariant function $e^x$ from the additive analysis [4] for the derivative of a fractional order is obtained from the Mittag-Leﬄer function using the following formula

$$h_\alpha(x) = \sum_{k=1}^{\infty} \frac{x^{-1+k\alpha}}{(-1 + k\alpha)!} = \frac{x^{\alpha-1}}{(\alpha - 1)!} + \frac{x^{2\alpha-1}}{(2\alpha - 1)!} + ...\quad (7)$$

Really, $D^\alpha h_\alpha(x) = D^\alpha \left[ \frac{x^{\alpha-1}}{(\alpha - 1)!} + \frac{x^{2\alpha-1}}{(2\alpha - 1)!} + ... \right] = \frac{x^{-1}}{(1-1)!} + \frac{x^{\alpha-1}}{(\alpha - 1)!} + \frac{x^{2\alpha-1}}{(2\alpha - 1)!} + ...\quad (7')$

Thus, as follows from (6), the first term of the latter is $\delta(E)$, that is, Dirac delta function. And the rest are $h_\alpha(x)$. If $x > 0$, then from (7') we have:

$$D^\alpha h_\alpha(x) = h_\alpha(x).\quad (8)$$

Therefore, we construct a new function that is an analogue of the Euler function $e^{\lambda x}$ for the fractional derivative starting from (7) in the following form:

$$h_\alpha(x,\lambda) = \sum_{k=1}^{\infty} \frac{\lambda^{k-1} x^{k\alpha-1}}{(k\alpha - 1)!} = \frac{x^{\alpha-1}}{(\alpha - 1)!} + \lambda \frac{x^{2\alpha-1}}{(2\alpha - 1)!} + \lambda^2 \frac{x^{3\alpha-1}}{(3\alpha - 1)!} + ...\quad (9)$$

Really:

$$D^\alpha h_\alpha(x,\lambda) = D^\alpha \sum_{k=1}^{\infty} \frac{\lambda^{k-1} x^{k\alpha-1}}{(k\alpha - 1)!} = \sum_{k=1}^{\infty} \lambda^{k-1} \frac{x^{(k-1)\alpha-1}}{[(k-1)\alpha - 1]!} = \frac{x^{-1}}{(1-1)!} + \lambda \frac{x^{\alpha-1}}{(\alpha - 1)!} + \lambda^2 \frac{x^{2\alpha-1}}{(2\alpha - 1)!} + ...,\quad (7')$$

Assuming that $x > 0$ ($\delta(x) \equiv 0$), starting from (6), we have:

$$D^\alpha h_\alpha(x,\lambda) = \lambda h_\alpha(x,\lambda).\quad (10)$$

Now, returning to the analysis or algebra [4], we show that for any real $\alpha \in (0,1]$ there are natural numbers $m$ and $n$ ($m, n \in N, m \leq n$) such that for any

$$\left| \alpha - \frac{2m + 1}{2n + 1} \right| < \varepsilon,\quad (11)$$

and these $m$ and $n$ are not the only ones.
Indeed, any real number can be arbitrarily closely approximated by a rational number, and any rational number can be arbitrarily closely approximated by a number of the form $2^{m+1}/2n+1$.

Let $\alpha \in (0, 1), \varepsilon = 10^{-k}, k \in N$, then the representation takes place

$$\alpha = 0, \alpha_1 \alpha_2 \ldots \alpha_{k-1} \alpha_k \alpha_{k+1}, \ldots,$$

where each $\alpha_i$ takes one of the values from zero to nine. Then we consider the following rational number:

$$\frac{\alpha_1 \alpha_2 \ldots \alpha_k \alpha_{k+1} \alpha_{k+2}}{10^{k+2}} \approx \frac{\alpha_1 \alpha_2 \ldots \alpha_{k+2} + q}{10^{k+2} + p},$$

where $p > 2$ is a prime number, but $q \in N$ is such that $(\alpha_1, \alpha_2, \ldots, \alpha_{k+2} + q)$ would be an odd number.

Thus, there exist such $m, n \in N, m < n$ that

$$\frac{\alpha_1 \alpha_2 \ldots \alpha_{k+2} + q}{10^{k+2} + p} = \frac{2m + 1}{2n + 1} \approx \alpha.$$

**Theorem.** Let $m, n \in N, m < n$, then the following formula holds

$$D^\frac{2m+1}{2n+1} h^i_\frac{1}{2n+1} \left( x, \lambda \frac{1}{2n+1+1} \right) = \lambda h^i_\frac{1}{2n+1} \left( x, \lambda \frac{1}{2n+1+1} \right).$$

**Remark.** It is easy to see that when $m = n = 0$ we have:

$$Dh_1 (x, \lambda) = \lambda h_1 (x, \lambda)$$

and

$$h_1 (x, \lambda) = e^{\lambda x}.$$

4. Conclusion

For the first time, the Mittag-Leffler function is represented by an exponential function. This allows us to analytically construct a solution of the Cauchy problem for linear differential operator equations of the fractional derivative. Due to the representation of the solution with the help of an exponential function, the asymptotic stability of the solution of the Cauchy problem is shown when the roots of the characteristic equation lie on the left half-plane.

**References**


ASYMPTOTICAL METHOD TO SOLUTION OF IDENTIFICATION PROBLEM FOR DEFINING THE PARAMETERS OF DISCRETE DYNAMICAL SYSTEM IN GAS-LIFT PROCESS

F.A. ALIEV\textsuperscript{1,2}, N.S. HAJIYEVA\textsuperscript{1}, A.A. NAMAZOV\textsuperscript{1}, N.A. SAFAROVA\textsuperscript{1}, N.SH. HUSEYNOVA\textsuperscript{1}

\textsuperscript{1}Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
\textsuperscript{2}Institute of Information Technology of ANAS, Baku, Azerbaijan
e-mail: f_aliev@yahoo.com, nazile.m@mail.ru, atif.namazov@gmail.com, narchis2003@yahoo.com, nargiz_huseynova@yahoo.com

1. Introduction

In the paper the asymptotical method for the identification problem to determine the parameters of discrete dynamical system in gas-lift process is considered. By means of the quasilinearization method [1], the nonlinear discrete equation is linearized. Further using the statistical data for the initial and final conditions for the equation the corresponding quadratic functional is constructed and its gradient is found. The computational algorithm for the solution of considered identification problem is offered. The results are illustrated on the example for finding the coefficient of hydraulic resistance in the tubes. It is shown that the statistical value of the coefficient of hydraulic resistance differ from the obtained value of the coefficient of hydraulic resistance on the order of $10^{-4}$.

2. Main problem

It is known that the equation of motion of flow of gas and gas-liquid mixture is in the form of nonlinear ordinary differential equation [2],

\begin{equation}
\dot{Q} = \frac{2a_1(\lambda_1)\rho_1 F_1}{c_1^2 \rho_1^2 F_1^2 \mu - Q^2} Q, \quad 0 \leq x < l - 0,
\end{equation}

\begin{equation}
\dot{Q} = \frac{2a_2(\lambda_2)\rho_2 F_2}{c_2^2 \rho_2^2 F_2^2 \mu - Q^2} Q, \quad l + 0 < x \leq 2l,
\end{equation}

\begin{equation}
Q(0) = u,
\end{equation}

where $\mu = \varepsilon^2$ and the parameters are defined as in [2]. The discrete nonlinear differential equations corresponding to the equations (1) and (2) have the following form

\begin{equation}
Q(i + 1) = Q(i) + h \frac{2a_1(\lambda_1)\rho_1 F_1}{c_1^2 \rho_1^2 F_1^2 \mu - Q^2(i)} Q(i), \quad 0 \leq i \leq N - 1,
\end{equation}

\begin{equation}
Q(i + 1) = Q(i) + h \frac{2a_2(\lambda_2)\rho_2 F_2}{c_2^2 \rho_2^2 F_2^2 \mu - Q^2(i)} Q(i), \quad N + 1 \leq i \leq 2N - 1,
\end{equation}

where $h$-sufficiently small number.

At the discrete point-$N$ the equations (4) and (5) are connected with each other as follows
\[ Q(N + 1) = \gamma Q(N) + \left(-\delta_3 (Q(N) - \delta_2)^2 + \delta_1\right) \tilde{Q}, \]

where the parameters are defined as in [4]. Let’s have some statistical data that at the given initial volumes of gas \( \tilde{Q}_s(0) \) the debit \( \tilde{Q}_s(2N) \) is measured at the output, i.e. \( \tilde{Q}_s(0) \) and \( \tilde{Q}_s(2N) \) are known \((s = 1, n)\), where \( s \) is the number of measurements.

It is required to find such values of the coefficient of hydraulic resistance \( \lambda_2 \) [3], which the equation (5) will describe the motion of GLM in the lift, closer to practice (adequate mathematical model). To solve this problem, it is required to minimize the functional

\[ J^k = \sum_{s=1}^{n} \left| Q^k_s(2N) - \tilde{Q}_s^k(2N) \right|^2. \]

Some nominal trajectory \( Q^0(i) \), the parameter \( a_0^2 \) are selected and it is assumed that \((k-1)\)-iteration has been already fulfilled. After linearizing equations (4) and (5) relatively these data, the obtained equations may be written at the end of the intervals, i.e. \( Q^k(2N) \) is written as follows

\[ Q^k(2N) = \left( (\Phi^0_3)^{k-1} + \mu(\Phi^1_3)^{k-1} \right) Q^k(N + 1) + \left( (\Phi^0_4)^{k-1} + \mu(\Phi^1_4)^{k-1} \right) a_2^k + \left( (\Phi^0_5)^{k-1} + \mu(\Phi^1_5)^{k-1} \right), \]

where

\[
(\Phi^0_3)^{k-1} = \prod_{i=2N+1}^{N+1} A^k_{20}(i), A^{k-1}_{20}(i) = E,
\]

\[
(\Phi^1_3)^{k-1} = \prod_{i=2N-2}^{N+1} A^{k-1}_{21}(N - 1) A^{k-1}_{20}(i), A^{k-1}_{21}(i) = h \frac{4a_2^{k-1} \rho_2^2 F_2^2 c_2^2}{(Q^{k-1})^3(i)},
\]

\[
(\Phi^0_4)^{k-1} = \sum_{j=N+2}^{2N-1} \left( \prod_{i=2N-1}^{j} A^{k-1}_{20}(i) \right) B^{k-1}_{20}(j - 1) + B^{k-1}_{20}(2N - 1),
\]

\[
(\Phi^1_4)^{k-1} = \sum_{j=N+2}^{2N-1} \left( \prod_{i=2N-1}^{j} A^{k-1}_{20}(i) \right) B^{k-1}_{21}(j - 1) + \sum_{j=N+2}^{2N-1} \left( \prod_{i=2N-2}^{j} A^{k-1}_{21}(2N - 1) A^{k-1}_{20}(i) \right)
\times B^{k-1}_{20}(j - 1) + B^{k-1}_{21}(2N - 1), B^{k-1}_{20}(i) = -2\rho_2 F_2 h, B^{k-1}_{21}(i) = -h \frac{2a_2^{k-1} \rho_2^2 F_2^3 c_2^2}{(Q^{k-1})^3(i)},
\]

\[
(\Phi^0_5)^{k-1} = \sum_{j=N+2}^{2N-1} \left( \prod_{i=2N-1}^{j} A^{k-1}_{20}(i) \right) C^{k-1}_{20}(j - 1) + C^{k-1}_{20}(2N - 1),
\]

\[
(\Phi^1_5)^{k-1} = \sum_{j=N}^{2N-1} \left( \prod_{i=2N-1}^{j} A^{k-1}_{20}(i) \right) C^{k-1}_{21}(j - 1) + C^{k-1}_{21}(2N - 1)
\times \prod_{i=2N-2}^{j} A^{k-1}_{21}(2N - 1) A^{k-1}_{20}(i) C^{k-1}_{20}(j - 1), C^{k-1}_{20}(i) = 0, C^{k-1}_{20}(i) = 0,
\]

\[
C^{k-1}_{21}(i) = 0. \]
\[ C_{21}^{k-1} (i) = -h \left( \frac{6a_{2}^{k-1} \rho_{2}^{3} F_{2}^{3} c_{2}^{2}}{(Q_{k-1}^{i})^{3}} + \frac{2(a_{2}^{k-1})^{2} \rho_{2}^{3} F_{2}^{3} c_{2}^{2}}{(Q_{k-1}^{i})^{3}} \right). \]

After putting (8) into (7), seeking \( a_{2}^{k} \) in the form

\[ a_{2}^{k} = a_{20}^{k} + \mu a_{21}^{k}, \quad (9) \]

we get the gradient of the functional and equate it to zero. Further we obtain the expressions for \( a_{20}^{k}, a_{21}^{k} \)

\[ a_{20}^{k} = -\sum_{s=1}^{n} \left( \left( \Phi_{4s}^{0} \right)^{k-1} \right)^{-1} \left( \left( \Phi_{3s}^{0} \right)^{k-1} \left( \Phi_{4s}^{0} \right)^{k-1} Q_{s}^{k} (N + 1) + \left( \Phi_{5s}^{0} \right)^{k-1} \left( \Phi_{4s}^{0} \right)^{k-1} \right) \]

\[ -\bar{Q}_{s}^{k} (2N) \left( \Phi_{4s}^{0} \right)^{k-1} = 0, \quad (10) \]

\[ a_{21}^{k} = -\sum_{s=1}^{n} \left( \left( \Phi_{4s}^{0} \right)^{k-1} \right)^{2} + 2\mu \left( \Phi_{4s}^{0} \right)^{k-1} \left( \Phi_{4s}^{0} \right)^{k-1} \left( \Phi_{3s}^{0} \right)^{k-1} Q_{s}^{k} (N + 1) \]

\[ +(\Phi_{3s}^{0} \left( \Phi_{4s}^{0} \right)^{k-1} - \bar{Q}_{s}^{k} (2N) \left( \Phi_{4s}^{0} \right)^{k-1} + \left( \Phi_{3s}^{0} \right)^{k-1} \left( \Phi_{4s}^{0} \right)^{k-1} Q_{s}^{k} (N + 1) + \left( \Phi_{5s}^{0} \right)^{k-1} (i) \left( \Phi_{4s}^{0} \right)^{k-1} \]

\[ -2\left( \Phi_{4s}^{0} \right)^{k-1} \left( \Phi_{3s}^{0} \right)^{k-1} \left( \left( \Phi_{4s}^{0} \right)^{k-1} \right)^{-1} \left( \left( \Phi_{3s}^{0} \right)^{k-1} \left( \Phi_{4s}^{0} \right)^{k-1} Q_{s}^{k} (N + 1) \right) \]

\[ + \left( \Phi_{3s}^{0} \right)^{k-1} \left( \Phi_{4s}^{0} \right)^{k-1} - \bar{Q}_{s}^{k} (2N) \left( \Phi_{4s}^{0} \right)^{k-1} \right) \]. \quad (11) \]

Putting the values of \( a_{20}^{k} \) and \( a_{21}^{k} \) from the equations (10) and (11) into (9), we get \( a_{2}^{k} \). For calculating the coefficient of hydraulic resistance \( \lambda_{2} \) we get the following formula [4],

\[ \lambda_{2} = \frac{4a_{2} D_{2}}{\omega_{2}} - \frac{2g D_{2}}{\omega_{2}}. \]

Let the parameters from equations (4)-(5) be given by

\[ l = 1485 \, m, \quad c = 331 \, m/san, \quad \rho = 700 \, kg/m^{3}, \quad \lambda = 0.01, \quad d = (114^{2} - 73^{2})^{1/2} \cdot 10^{-3} \, m \text{ for } 0 \leq i \leq N - 1; \quad c = 850 \, m/san, \quad \rho = 717 \, kg/m^{3}, \quad d = 0.073 \, m, \quad \lambda = 0.23 \, \text{ for } N + 1 \leq i \leq 2N. \]

After applying the above method we obtain that \( \lambda_{c} = 0.2353 \). Note that \( \lambda_{c} \) differs from \( \lambda_{c} \) (the hydraulic resistance value from practice) to the order \( 10^{-3} \), and it shows the efficiency of the proposed method.

**Keywords:** Nonlinear discrete equation, the method of quasilinearization, the gradient of the functional, identification, statistical data, the coefficient of hydraulic resistance.

**AMS Subject Classification:** 49J15, 49J35.

**References**


ON THE SOLVING OF SYLVESTER TYPE MATRIX EQUATION

F.A ALIEV1, V.B. LARIN2

1Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
2Institute of Information Technology of ANAS, Baku, Azerbaijan
3Institute of Mechanics of the Academy of Sciences of Ukraine, Ukraine, Kiev
e-mail: faliev@yahoo.com, vblarin@gmail.com

Abstract. A solution of two problems related to the matrix equation of Sylvester type is
given. In the first problem, the procedures for linear matrix inequalities are used to construct
the solution of this equation. In the second problem, when a matrix is given which is not a
solution of this equation, it is required to find such solution of the original equation, which most
accurately approximates the given matrix. For this, an algorithm for constructing a general
solution of the Sylvester matrix equation is used.

Keywords: Sylvester matrix equation, linear matrix inequalities, general solution.

AMS Subject Classification: 39B05, 39B42.

1. INTRODUCTION

In various problems of motion control an important place is occupied by questions connected
with the development of algorithms for solving matrix equations (see, for example, [4], and
references therein). Here it can be noted that the algorithms for constructing the solution of
the Sylvester matrix equations continue to attract the attention of researchers [5-7]. Thus, in
[7] two problems are considered for constructing a solution of a matrix equation of the Sylvester
type:

\[ AXD + CX^T D = E. \]  

(1)

In (1) the sought matrix \( X \) has the dimension \( m \times n \), the superscript "T" hereinafter means
transposition. In the first problem, we need to find the solution of (1). The second problem is
as follows. Let is given a matrix \( X_f \), which is not a solution of (1). It is necessary to find a
matrix \( X^* \), belonging to the set \( S_r \) of solutions of (1), which minimizes the following norm

\[ \|X^* - X_f\|_F = \min_{X \in S_r} \|X - X_f\|_F. \]  

(2)

Hereinafter \( \| \cdot \|_F \) means the Frobenius norm (Euclidean or spherical norm [6]).

Below the algorithms for solving of these problems will be considered. Thus, to solve the first
problem, an algorithm based on the use of procedures of linear matrix inequalities (LMI [2]) will
be considered, for solving the second problem the approach described in [1] will be used.
2. General relations of LMI [2]

As noted in [2] (relations (2.3), (2.4)), the matrix inequality:

\[
\begin{bmatrix}
Q(x) & S(x) \\
S^T(x) & R(x)
\end{bmatrix} > 0,
\]

(3)

where the matrices \( Q(x) = Q^T(x), R(x) = R^T(x), S(x) \) linearly depend on \( x \), equivalent to the following matrix inequalities:

\[
R(x) > 0, Q(x) - S(x)R^{-1}(x)S^T(x) > 0.
\]

(4)

Consider the following LMI:

\[
\begin{bmatrix}
Z & T \\
TT & I
\end{bmatrix} > 0, Z = Z^T,
\]

(5)

which, according to (3), (4), can be written in the form

\[
Z > TT^T.
\]

(6)

Hereinafter \( I \) is the unit matrix of the corresponding size. The relations (3) - (5) allow to consider the following standard LMI problem on eigenvalues (relations (2.9) (§2.2.2 [2])), namely, the problem of minimizing of a linear function \( cx \) (for example, \( cx = tr(Z) \), where \( tr(Z) \) is the trace of the matrix \( Z \)) under the conditions (5). To solve this problem, can be used the mincx.m procedure of the MATLAB package [3].

3. The solution of equation (1)

We use the above relations for solving the first problem. Thus, it is necessary to find a matrix \( X \), satisfying equation (1). Let in (5) \( T = AXB + CX^TD - E \). Using the procedure mincx.m of the MATLAB package, we minimize \( tr(Z) \) in inequality (6). For a sufficiently small value \( tr(Z) \approx 0 \), can be assumed that \( T \approx 0 \), and, consequently, the corresponding value of \( X \) is a solution of (1). However, to solve the second of the problems considered in [7], it is expedient to use an approach based on the procedure of the Kronecker product. As noted in [4], equation (1) can be represented as a system of linear algebraic equations:

\[
H vec(X) = vec(E).
\]

(7)

In (14) \( H = B^T \otimes A + (D^T \otimes C)P(m, n) \), where the symbol \( \otimes \) means the Kronecker product \( (A \otimes B = (a_{ij}B) \) see. [5]), and \( vec(X) = (x_{11}, x_{12}, \ldots, x_{m1}, x_{12}, x_{22}, \ldots, x_{m2}, \ldots, x_{mn})^T \in R^{mn} \). The matrix \( P(m, n) \) is defined as follows:

\[
P(m, n) = \begin{bmatrix}
P_{11}^T & P_{12}^T & \cdots & P_{1n}^T \\
P_{21}^T & P_{22}^T & \cdots & P_{2n}^T \\
\vdots & \vdots & \ddots & \vdots \\
P_{m1}^T & P_{m2}^T & \cdots & P_{mn}^T
\end{bmatrix} \in R^{mn \times mn}
\]

(8)

In (8) the matrices \( P_{ij} \) of dimension \( m \times n \), have the following structure: the element in the position \((i, j)\) is 1, all others are zeros. Denoting \( z = vec(X), b = vec(E) \), we rewrite system (7) as follows

\[
Hz = b.
\]

(9)

To construct the general solution of (9), one can use the approach of [1].
4. Algorithm for constructing the general solution (1) [1]

Thus, the problem of constructing a general solution of equation (1) reduces to the problem of constructing a general solution of the linear algebraic equation (9). Consequently, the condition for the existence of a solution of (1) can be formulated as follows. For the existence of the solution (1), the matrices $H \ [H \ b]$ must have the same rank [6] (to calculate the rank of the matrix one can use the rank.m procedure).

Let us perform the singular decomposition of the matrix $H$ (procedure svd.m):

$$H = USV^T. \tag{10}$$

In (10) $U, V$ are orthogonal matrices, $S$ is diagonal matrix, the first $r$ ($r$ is the rank of matrix $H$) elements of diagonal of which are not equal to zero. Let consider the matrix $U^TH = SV^T$.

In connection with the above structure of the matrix $S$, only the first $r$ rows of the matrix $U^TH$ will be non-zero. Denote by $A_g$ the matrix formed from the first $r$ rows of the matrix $U^TH$. Multiplying the left and right sides of equation (9) by the matrix $U^T$ and leaving only the first $r$ rows in both parts, we rewrite (9) as follows:

$$A_gz = b_u. \tag{11}$$

Here the vector $b_u$ is formed from the first $r$ components of the vector $U^Tb$. Note that appearing in (11) matrix $A_g$ is the matrix of full rang. Therefore, to determine the general solution (9), we can use the relations [1]:

$$z = A_g^T (A_gA_g^T)^{-1} b_u + N\xi, N = \left( I - A_g^T (A_gA_g^T)^{-1} A_g \right). \tag{12}$$

Here the first term on the right-hand side defines a particular solution of (9) having a minimal norm, $\xi$ is vector of free parameters, which determines the general solution of (9).

Let produce, similar to (10), the singular expansion of the matrix $N = U_nS_nV_n^T$.

Let the first $q$ diagonal elements of the matrix $S_n$ are not equal to zero. Consequently, the matrix $NV_n = U_nS_n$ will have only the first $q$ columns nonzero. We denote the matrix consisting of the first $q$ columns of the matrix $NV_n$ as $N_q$ (defining the zero subspace of the matrix $A_g$). We will rewrite relation (12) as follows:

$$z = A_g^T (A_gA_g^T)^{-1} b_u + N_q\xi_q, \tag{13}$$

where the dimension of the free parameters vector $\xi_q$ is $q$.

Having determined, according to (13), the vector $z$, i.e. the general solution (9) (having given in one way or another the vector $\xi_q$), then, using the procedure reshape.m, it is possible, according to the vector $z$, to construct a matrix $X$, defining the general solution of (1).

5. The second problem of [7]

Thus, as noted in the Introduction, the second problem of [7] is as follows. Suppose that are given a matrix $X_f$ which does not belongs to the set of solutions of equation (1). It is necessary to find a matrix $X^*$, belonging to the set of solutions of equation (1), which most accurately approximates the matrix $X_f$ (see the relation (2)).

Taking into account that for a matrix of dimension $n \times m$ (see §4.48 [6]) $\|A\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|^2$ it can be argued that $\|A\|_F = \|\text{vec}(A)\|_2^2$.

Thus, the problem is reduced to approximating of the vector $\text{vec}(X_f)$, by the vectors defined from (13), i.e. to the corresponding selection of the free parameters vector. In other words, it is necessary, by choosing the elements of the vector $\xi_q$, to minimize the discrepancy of the following linear system:

$$N_q\xi_q = b_q, b_q = \text{vec}(X_f) - A_g^T (A_gA_g^T)^{-1} b_u. \tag{14}$$
The corresponding solution of system (14) has the form (see §15.43 [5]):

$$\xi_q = (N_q^T N_q)^{-1} N_q^T b_q.$$  \hspace{1cm} (15)

Note that the procedure of solving (15) is realized by the procedure “\” of the MATLAB package.

Thus, the expression for the matrix $X^*$, which most accurately approximates the matrix $X_f$ has the form

$$\text{vec}(X^*) = A_q^T (A_q A_q^T)^{-1} b_u + N_q \xi_q = z_0 + N_q \xi_q,$$  \hspace{1cm} (16)

where the vector $\xi_q$ is defined by (15).

6. Conclusion

A solution of the problems considered in [7], connected with the solution of the Sylvester type matrix equation, is given. In the first problem to construct the solution of this equation the procedures for linear matrix inequalities are used [2]. In the second problem is used the algorithm [1] for construct a general solution of the Sylvester matrix equation. The effectiveness of the proposed approaches is illustrated by examples.

References

NUMERICAL-ANALYTICAL METHOD FOR SOLVING OF THE FIRST ORDER PARTIAL QUASI-LINEAR EQUATIONS

F.A. ALIEV\textsuperscript{1}, N.A. ALIEV\textsuperscript{1}, R.M. TAGIEV\textsuperscript{1}, Y.V. MAMMADOVA\textsuperscript{1}, M.F. RAJABOV\textsuperscript{2}

\textsuperscript{1}Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
\textsuperscript{2}Institute of Control Systems of ANAS, Baku, Azerbaijan
e-mail: f.aliev@yahoo.com, tagiyev.reshad@gmail.com

Abstract. Two-dimensional first order non-linear (quasi-linear) partial differential equation, which arises in the solving the system of hyperbolic equations describing the oil production by gas lift using the sweep method, is considered. By using the method of characteristics is shown that the seeking solution can be defined from the corresponding implicit algebraic relation by the help of the fixed point method. The computational algorithm for solving the implicit algebraic relation is giving. For the particular case, when the initial function is constant the analytical solution of the partial quasi linear equation is found.

Keywords: Partial differential equation, quasi linear equation, method of characteristics, implicit solution.

AMS Subject Classification: 35F20, 35J05, 65M25.

1. Introduction

For ordinary linear differential equations with constant coefficients the general solution can be determined on the basis of Euler’s theory, where the finding of the explicit representation both of Cauchy problem [2] and the boundary problem is not difficult. For the problems of the ordinary differential equations with variable coefficients, the solutions are obtained in an explicit form [2]. It should be noted that such studies for partial differential equations are almost neglected.

As is known, the solution of the Cauchy problem for the second order differential equation of hyperbolic type with constant coefficients is given explicitly by d’Alembert’s (in the two dimensions case), Poisson’s (in the three dimensions case) and Kirchhoff’s (in the case of four dimensions) formulas [3]. The solution of the mixed problem for these equations is, generally, represented as the series [7, 8]. Considering the presentation in the explicit form of the solution of the problem for the partial linear differential equations with constant and variable coefficients, only narrow class of such problems are covered.

In this work one example of the first order quasi-linear partial differential equation, which arises in the mathematical modeling of the oil production by the gas lift method is considered [6]. Based on the method of characteristics the representation of the solution of Cauchy problem for this equation is given. The results are illustrated by the example arising from the specific practical problems [1, 4].

2. Statement of the problem

Let the first order quasi-linear partial differential equation in the form

$$\frac{\partial S(x,t)}{\partial x} + F S(x,t) \frac{\partial S(x,t)}{\partial t} - 2aS(x,t) = 0, \quad x \in (0,l), \quad t > 0,$$

(1)
with the initial condition
\[ S(x,0) = \varphi(x), \quad x \in [0,l], \]  
(2)
where \( F \) and \( a \) are real constants, \( \varphi(x) \) is the known continuous real-valued function, and \( S(x,t) \) is the seeking function, be given.

The solution of equation (1) will be sought in an implicit form
\[ \chi(x,t,S(x,t)) = 0. \]  
(3)
Then differentiating (3) both by \( x \) and \( t \), we’ll have:
\[ \frac{\partial\chi}{\partial x} + \frac{\partial\chi}{\partial S} \frac{\partial S}{\partial x} = 0, \quad \frac{\partial\chi}{\partial S} \frac{\partial S}{\partial t} = 0, \]
or
\[ \frac{\partial S(x,t)}{\partial x} = -\frac{\partial\chi}{\partial x}, \quad \frac{\partial S(x,t)}{\partial t} = -\frac{\partial\chi}{\partial S}. \]  
(4)
Substituting (4) into (1) we obtain:
\[ \frac{\partial\chi}{\partial x} + FS \frac{\partial\chi}{\partial S} - 2aS = 0, \]
or
\[ \frac{\partial\chi}{\partial x} + FS \frac{\partial\chi}{\partial t} + 2aS \frac{\partial\chi}{\partial S} = 0, \]  
(5)
which is the first order linear partial differential equation for the function \( \chi(x,t,S(x,t)) \) from (3).

3. The method of characteristics
Let us define the characteristics of the equation (5):
\[ \frac{dx}{dt} = \frac{dS}{2aS} \]
from which we obtain the following equations:
\[ dx = \frac{dS}{2aS} \quad \text{and} \quad dt = dS \quad \frac{2a}{FS}, \]
or
\[ \frac{dS}{S} = 2adx, dS = \frac{2a}{FS} dt. \]
After integration of the last we define the following characteristics or a functional-invariant Yerugin’s solution [5]
\[ \begin{align*}
\ln S(x,t) - 2ax &= C_1, \\
S(x,t) - \frac{2at}{F} &= C_2.
\end{align*} \]  
(6)
Here \( C_1 \) and \( C_2 \) are constants with which the characteristics are defined. Taking \( t = 0 \) in (6) we have:
\[ \begin{align*}
\ln S(x,0) - 2ax &= C_1, \\
S(x,0) &= C_2.
\end{align*} \]
Taking into account the boundary condition (2), we obtain:
\[ \begin{align*}
\ln \varphi(x) - 2ax &= C_1, \\
\varphi(x) &= C_2,
\end{align*} \]
or from the second equation we define
\[ x = \varphi^{-1}(C_2), \]
and substituting it into the first equation before the past, we find
\[ \ln C_2 - 2a\varphi^{-1}(C_2) = C_1, \]
Substituting (8) and (9) into (1) we have:

or

or

Similarly

Finally, taking into account (6) from the last, we have:

or

It is easy to see that (7) satisfies the initial condition (2). Now we show that (7) satisfies the equation (1), too.

Indeed,

or

Similarly

Substituting (8) and (9) into (1) we have:

\[\frac{\varphi'(FS - 2at)}{FS^2 - 2atS - \varphi t} + \frac{FS}{FS^2 - 2atS - \varphi t} = \frac{\varphi'(FS - 2at)}{FS^2 - 2atS - \varphi t},\]

i.e., the implicit function \(S(x,t)\), given in the form (7), is a solution of the equation (1).

REFERENCES

LIMIT THEOREMS FOR THE FAMILY OF THE FIRST PASSAGE TIME OF THE PARABOLA BY A RANDOM WALK DESCRIBED BY THE AUTOREGRESSION PROCESS OF ORDER ONE (AR(1))

S.A. ALIEV¹, F.H. RAHIMOV², T.E. HASHIMOVA¹, A.D. FARHADOVA², G.A. BAGIROVA³

¹Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
²Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
³Baku State University, Baku, Azerbaijan
e-mail: soltan.aliyev@imm.az, ragimovf@rambler.ru, farxadovaaynura@yahoo.com, gulnare_agayeva@mail.ru

Abstract. In the paper are proved limit theorems of the family of the first passage time of the parabola by a random walk described by the autoregression process of order one AR(1).

Keywords: Autoregression process of order one AR(1), random walk, first passage time.

AMS Subject Classification: 60F05.

1. Introduction

Let on the probability space $(\Omega, \mathcal{F}, P)$ the sequence of independent identically distributed random variables $\xi_n$, $n \geq 1$ be given. It is well known that the autoregression process of order one is determined by the equality

$$X_n = \beta X_{n-1} + \xi_n, \quad n \geq 1,$$

where $\beta$ is some fixed number and the initial value of the process $X_0$ is independent on the innovation $\{\xi_n\}$ [3].

Denote $T_n = \sum_{k=1}^{n} X_k X_{k-1}$, $n \geq 1$ and consider the family

$$\tau_n = \inf \{n \geq 1 : T_n \geq a\sqrt{n}\}$$

of first passage time of the parabola $f_a(t) = a\sqrt{t}$ by the Markov random walk $T_n$, $n \geq 1$.

Note that in the case when the $T_n$ is formed by the sums of independent identically distributed random variables, i.e. $T_n = S_n = \sum_{k=1}^{n} \xi_k$, there are a various works (see e.i. [1, 13]) devoted to study of the family of stopping time of the form (1) for rather wide class of nonlinear boundarys $f_a(t)$.

At present a great attention is paid to study of boundary problems for Markov’s random walks to the class of which the ordinary random walks $S_n = \sum_{k=1}^{n} \xi_k$ ([2, 4, 5, 12]) belong.
Limit theorems for the family of the first passage time of the evel (of the linear boundary) \((f_a(t) = a)\) by the Markov random walk \(T_n, n \geq 1\) were studied in the paper \([6, 8]\).

Boundary problems related to the passage of nonlinear boundaries \((f_a(t) \neq a)\) by the Markov random walk were studied significantly less. The first passage time of the AR(1) autoregression process beyond nonlinear boundary was first considered in the paper \([5]\) for the case when innovation is generated by exponential distribution.

In the present paper we prove limit theorems for the family of first passage time of the form (1).

2. FORMULATION OF THE MAIN RESULT

First of all we note that some asymptotic properties of the process \(T_n = \sum_{k=1}^{n} X_kX_{k-1}, n \geq 1\) were studied in the papers \([8–11]\).

We give the following known asymptotic properties of the process \(T_n\) that we will need in the sequel.

In the paper \([2]\) (see also \([7]\) chapter VII, 1 p., 174) it is shown that subject to the conditions \(E|X_0|^2 < \infty, |\beta| < 1, E\xi_1 = 0\) and \(D\xi_1 = 1\) the following limit relations are valid:

\[
\frac{T_n}{n} \xrightarrow{a.s.} \lambda = \frac{\beta}{1 - \beta^2} \quad \text{as } n \to \infty \tag{2}
\]

and

\[
\lim_{n \to \infty} P\left( \frac{T_n - n\lambda}{\sqrt{n}} \leq x \right) = \phi(x\theta), \tag{3}
\]

where \(\theta = \sqrt{1 - \beta^2}\) and \(\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy\).

**Theorem 1.** Let the following conditions be satisfied

\[E|X_0|^2 < \infty, \quad 0 < |\beta| < 1, \quad E\xi_1 = 0\] and \(D\xi_1 = 1\).

Then

\[\frac{\tau_a}{N_a} \xrightarrow{a.s.} 1 \quad \text{as } a \to \infty,\]

where \(N_a = (a/\lambda)^2, \lambda = \frac{\beta}{1 - \beta^2}\).

**Theorem 2.** Let the conditions of Theorem 1 be satisfied. Then

\[\lim_{a \to \infty} P\left( \frac{\tau_a - N_a}{\sqrt{N_a}} \leq x \right) = \phi\left( \frac{|\beta| x}{2\theta} \right), \quad x \in R.\]

For proving these results we need the following facts formulated as lemmas.

**Lemma 1.** Let the sequence of random variables \(\eta_n, n \geq 1\) be uniformly continuous in probability, i.e. the relation be satisfied

\[\lim_{\delta \to 0} \sup_n P\left( \max_{1 \leq k \leq n} |\eta_{n+k} - \eta_n| > \varepsilon \right) = 0 \tag{4}\]

for any \(\varepsilon > 0\).

Let \(t_a, a > 0\) be the family of integer-valued random variables such that

\[\frac{t_a}{a} \xrightarrow{P} c, \quad a \to \infty,\]

where \(c > 0\) is some number.
Then, if $Y_n \overset{d}{\to} Y$, then $Y_{t_n} \overset{d}{\to} Y$ as $n \to \infty$, where the sign $\overset{d}{\to}$ means convergence in distribution.

This lemma is one of the variants of Anscombe’s known theorem [1,13].

**Lemma 2.** Let the sequences $\eta_n^{(1)}$ and $\eta_n^{(2)}$, $n \geq 1$ be uniformly continuous in probability, i.e. (2.3) is fulfilled for them. Then the sequence of the sums $\eta_n^{(1)} + \eta_n^{(2)}$, $n \geq 1$ is also uniformly continuous in probability.

Furthermore, if the sequences $\eta_n^{(1)}$ and $\eta_n^{(2)}$, $n \geq 1$ are stochastically bounded, then the sequence of products $\eta_n^{(1)} \eta_n^{(2)}$, $n \geq 1$ is uniformly continuous in probability.

This lemma is a special case of Lemma 1.4 from the monograph [13].

Note that if the sequence of random variables $\eta_n$, $n \geq 1$ converges almost surely to finite limit, then it is uniformly continuous in probability, i.e. (4) is fulfilled for it [13].

**Lemma 3.** In the conditions of Theorem 1, the sequence

$$T_n = T_n - m_A, \quad n \geq 1$$

is uniformly continuous in probability.

Statement of this lemma was proved in [2].

**Lemma 4.** In the conditions of Theorem 1, the sequence

$$\frac{X_n X_{n-1}}{\sqrt{n}} \rightarrow P$$

as $n \to \infty$.

**References**


LIGHT-CONE DISTRIBUTION AMPLITUDES OF LIGHT $J^{PC} = 2^{--}$
TENSOR MESONS IN QCD

T.M. ALIEV$^1$, S. BILMIS$^1$, K.CH. YANG$^2$

$^1$ Department of Physics, Middle East Technical University, Ankara, Turkey
$^2$ Department of Physics and Center for High Energy Physics, Chung Yuan Christian University, Taoyuan, Taiwan
e-mail: taliev@metu.edu.tr

We present a study for two-quark light-cone distribution amplitudes for light $J^{PC} = 2^{--}$ tensor meson states. We estimate the relevant parameters, the decay constants $f_T$ and $f_T^\perp$, and first Gegenbauer moment $a_1^\perp$, by using the QCD sum rule method.

The mass spectra of the negative parity tensor mesons $2^{--}$, containing light-light, light-heavy and heavy-heavy quarks, were calculated in [4] in the QCD sum rule approach. However, the light $2^{--}$ meson states, except $K_2(1820)$ meson, have not been observed yet. In this work, we will focus on the study of the light-cone distribution amplitudes (LCDAs) of the light $2^{--}$ meson states.

The rest of the paper is organized as follows. Firstly, we define light-cone distribution amplitudes for the $2^{--}$ meson states and discuss their properties and the parameters $f_T$, $f_T^\perp$ and $a_1^\perp$ are estimated within QCD sum rules method.

We define the chiral-even light-cone distribution amplitudes of a light tensor meson with quantum number $J^{PC} = 2^{--}$

\begin{eqnarray}
\langle P, \lambda | q_1(z) \gamma_\mu \gamma_5 q_2(-z) | 0 \rangle &=& i f_T m_T^3 \int_0^1 du e^{i(u-\bar{u})} \left\{ p_\mu \epsilon^{(\lambda)\alpha\beta}_{\alpha\beta} z^\alpha z^\beta (p_z)^2 \phi_1(u) \\
+ \epsilon^{(\lambda)}_{\perp \alpha \alpha} z^\alpha g_a(u) - \frac{1}{2} z_\mu \epsilon^{(\lambda)\alpha\beta}_{\alpha\beta} (p_z)^3 m_T^2 g_3(u) \right\}, \\
\langle P, \lambda | q_1(z) \gamma_\mu \gamma_5 q_2(-z) | 0 \rangle &=& i f_T m_T^3 \int_0^1 du e^{i(u-\bar{u})} e_{\mu\nu\alpha\beta} z^\nu p_\alpha \epsilon^{(\lambda)\beta\delta}_{\beta\delta} (p_z) g_v(u), \\
\langle P, \lambda | \bar{q}_1(z) \sigma_{\mu\nu} \gamma_5 q_2(-z) | 0 \rangle &=& f_T^\perp m_T^2 \int_0^1 du e^{i(u-\bar{u})} \left\{ \epsilon^{(\lambda)}_{\perp \alpha \alpha} p_\nu - \epsilon^{(\lambda)}_{\perp \alpha \alpha} p_\mu \right\} \frac{1}{p_z} \phi_1(u) + \left( p_\mu z_\nu - p_\nu z_\mu \right) \frac{m_T^2 \epsilon^{(\lambda)\alpha\beta}_{\alpha\beta} z^\alpha z^\beta (p_z)^3}{(p_z)^3} h_t(u) \\
+ \frac{1}{2} \left[ \epsilon^{(\lambda)}_{\perp \mu} z^\nu - \epsilon^{(\lambda)}_{\perp \nu} z^\mu \right] m_T^2 h_3(u), \\
\langle P, \lambda | \bar{q}_1(z) \gamma_5 q_2(-z) | 0 \rangle &=& f_T^\perp m_T^4 \int_0^1 du e^{i(u-\bar{u})} \frac{\epsilon^{(\lambda)\alpha\beta}_{\alpha\beta} z^\alpha z^\beta}{p_z} h_p(u),
\end{eqnarray}
where \( u \) and \( \bar{u} \equiv 1 - u \) are the momentum fractions carried by \( q_1 \) and \( \bar{q}_2 \) quarks, respectively, in the meson. \( \phi_{\|}, \phi_{\perp} \) are leading twist-2 LCDAs, \( g_v, g_a, h_t, h_p \) are twist-3 ones, and \( g_3 \) and \( h_3 \) are of twist-4. Here \( z_\mu \) and \( p_\nu \equiv P_\nu - m_T^2/(2p_\nu) \) are the two light-like vectors, with \( P_\nu \) and \( m_T \) being the momentum and the mass of the tensor meson, respectively and

\[
e_\perp^{(\lambda)} \frac{z_\nu}{p_\nu} = e_\perp^{(\lambda)} \frac{z_\nu}{p_\nu} - \frac{e_\perp^{(\lambda)} z_\nu}{p_\nu} \left( p_\mu - \frac{m_T^2}{2p_\nu} z_\mu \right). \tag{4}
\]

Using the conformal basis, the leading-twist LCDAs \( \phi_{\|,\perp}(u, \mu) \) can be expressed in a series of Gegenbauer polynomials. The LCDAs can be approximately expanded up to the term including the first Gegenbauer moment, \( a_1^{\|,\perp} \), as

\[
\begin{align*}
\phi_{\|}(u) &= 30u(1 - u)(2u - 1)\frac{3}{5}a_1^\|, \\
\phi_{\perp}(u) &= 6u(1 - u) + 30u(1 - u)(2u - 1)\frac{3}{5}a_1^\perp.
\end{align*} \tag{5, 6}
\]

Now we will use QCD sum rule approach to estimate the relevant parameters: the decay constants \( f_T \) and \( f_T^\perp \), and first Gegenbauer moment \( a_1^\perp \). For this aim first, we consider

\[
\Pi'_{\mu\nu;\alpha\beta} = i \int d^4x (0|j^{\|}_{\mu\nu;\alpha\beta}(x)j_{\alpha\beta}(y)|0)|_{y \to 0} e^{iqx},
\]

where \( j_{\alpha\beta} \) is the interpolating current,

\[
\begin{align*}
j_{\alpha\beta} &= \left[ \bar{q}_1(y)\gamma_\alpha \gamma_5 \vec{D}_\alpha q_2(y) + \bar{q}_1(y)\gamma_\beta \gamma_5 \vec{D}_\beta q_2(y) \right], \tag{8}
\end{align*}
\]

and

\[
\begin{align*}
j^{\|}_{\mu\nu;\alpha\beta} &= \bar{q}_2(x)\sigma_{\mu\nu} \gamma_5 \vec{D}_\alpha q_1(x). \tag{9}
\end{align*}
\]

The covariant derivative is defined as

\[
\begin{align*}
\vec{D}_\alpha &= \vec{D}_\alpha - \vec{A}_\alpha \\
&= i\gamma_\alpha \lambda^a A^a_\mu \vec{D}_\mu - i\gamma_\alpha \lambda^a A^a_\mu, \tag{10}
\end{align*}
\]

where \( \lambda^a \) are the Gell-Mann matrices. Omitting the details of calculations (see for example [2]) for the coefficient of the Lorentz structure \( [(g_{\mu\alpha}g_{\delta\beta} + g_{\mu\beta}g_{\delta\alpha})q_\nu - (\mu \leftrightarrow \nu)] \) in phenomenological part of the correlation function we get

\[
\Pi' = \frac{f_T f_T^\perp m_T^5}{(m_T^2 - q^2)} + \ldots. \tag{11}
\]

The choice of this structure is dictated by the fact that this structure contains only tensor meson contributions. To calculate the first Gegenbauer moment \( a_1^\perp \) that gives corrections to asymptotic form of \( \phi_{\perp} \), we consider the second correlation function introduced by

\[
\Pi_{\mu\nu;\alpha\beta} = i \int d^4x (0|j^{\|}_{\mu\nu;\alpha\beta}(x)j_{\alpha\beta}(y)|0)|_{y \to 0} e^{iqx}, \tag{12}
\]

where \( j^{\|}_{\mu\nu;\alpha\beta} \) is given by

\[
\begin{align*}
j^{\|}_{\mu\nu;\alpha\beta} &= \bar{q}_2(x)\sigma_{\mu\nu} \gamma_5 i \vec{D}_\delta q_1(x), \tag{13}
\end{align*}
\]

and satisfies

\[
\begin{align*}
\langle 0|j^{\|}_{\mu\nu;\alpha\beta}|T(q_1, \lambda)\rangle &= \frac{3}{5} a_1^\perp f_T f_T^\perp m_T^2 (e_\perp^{(\lambda)} q_\nu - e_\perp^{(\lambda)} q_\mu). \tag{13}
\end{align*}
\]

Using these definitions and separating the coefficient of the structure \( [(g_{\mu\alpha}g_{\delta\beta} + g_{\mu\beta}g_{\delta\alpha})q_\nu - (\mu \leftrightarrow \nu)] \), we get the following expression for the correlation function in terms of hadronic degrees of
freedom,

$$\Pi_{\mu\nu\alpha\beta} = \frac{f_T^+ m_T^2 f_T^+ m_T^2 (a_1^+)}{(m_T^2 - q^2)} \times \left( (g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})q_\nu - (\mu \leftrightarrow \nu) + \text{Other structures} \right). \quad (14)$$

The correlation functions can also be calculated in terms of quark-gluon fields using the operator product expansion (OPE). Using expressions of interpolating current $j_\mu$ and $j_\alpha\beta$ and for the correlation function, after lengthy calculation we get the following sum rules:

$$f_T^+ f_T^+ m_T^5 e^{-m_T^2/M^2} = \frac{\langle g_s^2 G^2 \rangle (m_{q_1} + m_{q_2})}{192\pi^2}, \quad (15a)$$

$$\frac{3}{5} a_1^+ f_T^+ f_T^+ m_T^5 e^{-m_T^2/M^2} = \frac{1}{192\pi^2 M^2} \left[ 2M^2 \langle g_s^2 G^2 \rangle (m_{q_1} - m_{q_2}) \ln \frac{\Lambda^2}{M^2} ight. \right.$$  

$$+ \langle g_s^2 G^2 \rangle \left[ (1 + 6\gamma_E)M^2 (m_{q_1} - m_{q_2}) - 24\pi^2 (\langle \bar{q}_1 q_1 \rangle - \langle \bar{q}_2 q_2 \rangle) \right]$$

$$- 24M^2 \left[ M^4 (m_{q_1} - m_{q_2}) - 4\pi^2 m_0^2 (\langle \bar{q}_1 q_1 \rangle - \langle \bar{q}_2 q_2 \rangle) \right] \right]. \quad (15b)$$

After subtracting the contributions of higher states and continuum, i.e., by doing the replacement:

$$(M^2)^n \rightarrow \frac{1}{\Gamma(n)} \int_0^{s_0} e^{-s/M^2} s^{n-1} ds,$$  

one can easily obtain two sum rule results for the determination of $f_T^+$ and $a_1^+$ from Eqs. (15a) and (15b).

For determination of $f_T^+$ and $a_1^+$, we need to know $f_T$. This value for light tensor mesons is determined in [1, 3], for this reason we do not present its expression here. Using the values of input parameters and $f_T$ our sum rules leads to the following values for $f_T^+$ and $a_1^+$:

$$f_T^+ = \begin{cases} 
8.2 \pm 2.2 \times 10^{-4}, & \text{for } \bar{s}s, \\
5.2 \pm 1.2 \times 10^{-4}, & \text{for } \bar{q}s, \\
3.5 \pm 1.1 \times 10^{-5}, & \text{for } \bar{q}q.
\end{cases} \quad (17)$$

$$a_1^+ = (48 \pm 12), \text{ for } \bar{q}s. \quad (18)$$

Here and for the results in the following calculations, the errors correspond to the uncertainties due to variation of input parameters, threshold for higher states $s_0$, theoretically predicted values for masses of tensor mesons, and Borel mass.

Keywords: Gegenbauer polynomials, light cone vectors, distribution amplitudes, twists of operators.

AMS Subject Classification: 12Yxx.

References


AN INVESTIGATION OF A MIXED PROBLEM FOR THE PARABOLIC EQUATION SECOND-ORDER

A.M. ALIEV¹, N.A ALIEV¹

¹Baku State University, Institute of Applied Mathematics, Baku, Azerbaijan
e-mail: aahmad07@rambler.ru

ABSTRACT. The paper considers a mathematical model of the process of oil movement to the central imperfect well in strata with relaxing porosity. In this case we obtain a mixed problem for an integrals-differential equation of parabolic type. Here the right-hand side of the boundary condition is a discontinuous function.

Keywords: Mixed problem, Boundary conditions, perfect well.

AMS Subject Classification: 35J05, 35M13.

1. Introduction

The work is devoted to the investigation of the solution of a mixed problem in a three-dimensional domain for a second-order parabolic differential equation. A non-standard approach to the solution of the problem arising during oil production is proposed. We note that the boundary conditions are given by means of a discontinuous function [1-4]. The study of the solution of this problem has a certain practical interest. The work considers a mathematical model of the process of oil flow to a central perfect well in a circular reservoir with a constant power.

2. Formulation of the problem

Let \( P = P(r, z, t) \), \( r \in (r_c, r_k) \), \( z \in (0, h) \), \( t > 0 \), \( r_c, R_k, h > 0 \) and real numbers, \( r_c < r_k \).

It is required to find the solution of the equation:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial P}{\partial r} \right) + \frac{\partial^2 P}{\partial z^2} = \chi \frac{\partial P}{\partial t}, \quad t > 0, \quad z \in (0, h), \quad r \in (r_c, r_k),
\]

satisfying the initial:

\[
P = P_{n,t} = \rho_h g L, \quad t = 0, \quad r \in [r_c, r_k], \quad z \in [0, h]
\]

and boundary conditions:

\[
\frac{\partial P}{\partial z} = 0, \quad z = 0, \quad z = h, \quad r \in [r_c, r_k],
\]

\[
\left\{ \begin{array}{ll}
P = P_k(t), & r = r_k, \quad t \geq 0, \quad z \in [0, h], \n\frac{\partial P}{\partial r} = \frac{\mu a}{2kz h}, & r = r_c, \quad t \geq 0, \quad z \in [0, h],
\end{array} \right.
\]

where \( P = P(r, z, t) \) is the required current pressure, \( \chi \) - the coefficient of piezoelectric conductivity, \( P_{rp} \) - the reservoir pressure, \( \rho_h \) - the density of the oil, \( t \) - the time, \( r \) and \( z \) respectively, the radial and vertical coordinates.
Separating the variable \( z \) from the variables \( r \) and \( t \) in the form:

\[
P(r, z, t) = Z(z)Q(r, t)
\]

from equation (1) we have:

\[
Z(z) \cdot \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial Q}{\partial r} \right) + Z''(z)Q(r, t) = \chi Z(z) \frac{\partial Q}{\partial t}.
\]

Dividing both parts of the resulting expression by the function \( Z(z)Q(r, t) \), we get:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial Q}{\partial r} \right) + \frac{Z''(z)}{Z(z)} = \chi \frac{\frac{\partial Q}{\partial t}}{Q}.
\]

or

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial Q}{\partial r} \right) - \chi \frac{\frac{\partial Q}{\partial t}}{Q} = -\frac{Z''(z)}{Z(z)} = \nu^2.
\]

Thus, after dividing the variables \( z, r, t \) in equation (1) with (5), we arrive at the following one-dimensional and two-dimensional equations:

\[
Z''(z) + \nu^2 Z(z) = 0
\]

\[
-\chi \frac{\frac{\partial Q}{\partial t}}{Q} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial Q}{\partial r} \right) - \nu^2 Q = 0
\]

In order to separate the variables in the boundary conditions (3), substituting (5) in (3), we obtain for the equation (7) the following two boundary conditions in the form:

\[
Z'(0) = Z'(h) = 0.
\]

Now we consider problem (7), (9). It is easy to see that the general solution of this problem has the form:

\[
Z(z) = C_1 \cos \nu z + C_2 \sin \nu z.
\]

Further, calculating the derivative of this function, we obtain:

\[
Z'(z) = -C_1 \nu \sin \nu z + C_2 \nu \cos \nu z.
\]

Substituting the value of this function into the first of the conditions (9), we obtain:

\[
C_2 \nu = 0 \quad \Rightarrow \quad C_2 = 0.
\]

In the same way, from the second condition appearing in (9), we get:

\[
-C_1 \nu \sin \nu h = 0 \Rightarrow \sin \nu h = 0, \nu_m h = m\pi.
\]

The eigenvalues and eigenfunctions of the spectral problem (7), (9) are obtained, respectively, in the form:

\[
\nu_m = \frac{m\pi}{h}, \quad m \in Z,
\]

\[
Z_m(z) = C_1 \cos \frac{m\pi}{h} z, \quad m \geq 0.
\]

Now we normalize these proper functions:

\[
\int_0^h Z_m^2(z) dz = C_1^2 \int_0^h \cos^2 \frac{m\pi}{h} z dz = C_1^2 \int_0^h \frac{1 + \cos \frac{2m\pi}{h} z}{2} dz = \frac{C_1^2}{2} h = 1,
\]

From here:

\[
C_1^2 = \frac{2}{h}, \quad C_1 = \sqrt{\frac{2}{h}}.
\]
Thus, we obtain the eigenvalues and functions of problem (7), (9) in the following form:

$$\nu_m = \frac{m\pi}{h}, Z_m(z) = \sqrt{\frac{2}{h}} \cos \left( \frac{m\pi z}{h} \right), m \geq 0.$$  \hfill (11)

Now, returning to the problem (1) - (4), we seek its solution in the form of a Fourier series:

$$P(r, z, t) = \sum_{m=0}^{\infty} Q_m(r, t) Z_m(z).$$ \hfill (12)

Substituting this expression into equation (1), we obtain:

$$\sum_{m=0}^{\infty} Z_m(z) \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial Q_m}{\partial r} \right) + \sum_{m=0}^{\infty} Q_m(r, t) Z_m''(z) = \chi \sum_{m=0}^{\infty} \frac{\partial Q_m(r, t)}{\partial t} Z_m(z),$$

Using (7) and bearing in mind that

$$Z_m''(z) = -\nu_m^2 Z_m(z)$$

and $$Z_m(z)$$ are linearly independent, then for $$Q(r, t)$$ the following equation is obtained:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial Q_m}{\partial r} \right) - \nu_m^2 Q_m(r, t) = \chi \frac{\partial Q_m(t)}{\partial t}, m \geq 0 \hfill (13)$$

To determine the conditions that are added to equation (13), write (12) in (2) and (4). After that, to determine, $$Q_m(r, t)$$ we get the following initial:

$$Q_m(r, 0) = \int_0^h P_{n,l}(z, r) Z_m(z) dz, m \geq 0,$$ \hfill (14)

and the boundary conditions:

$$Q_m(r_k, t) = \int_0^h Q_{k}(t, r_k, z) Z_m(z) dz, \quad m \geq 0,$$

$$r \frac{\partial Q_m}{\partial r} \bigg|_{r=r_c} = \int_0^h \frac{\mu q_k}{2\pi n} Z_m(z) dz, \quad m \geq 0.$$ \hfill (15)

Now, applying the Laplace transform to the mixed problem (13) - (15), we obtain:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \int_0^\infty e^{-\lambda t} Q_m(r, t) dt \right) - \nu_m^2 \int_0^\infty e^{-\lambda t} Q_m(r, t) dt = \chi \int_0^\infty e^{-\lambda t} \frac{\partial Q_m}{\partial t} dt.$$

After this, the relation $$Q_m(r, t)$$ is obtained as the inverse Laplace transform in the form:

$$Q_m(r, t) = \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda t} \tilde{Q}_m(r, \lambda) d\lambda.$$

Further from (12) is determined $$P(r, z, t).$$

**References**

ON A COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

AREEN ALKHATEEB

1Department of Mathematics, Eastern Mediterranean University, Famagusta, T.R. North Cyprus, via Mersin 10, Turkey
e-mail: areen.alkhateebv@emu.edu.tr

Fractional differential equations have gained considerable importance due to their varied applications in many problems of physics, chemistry, biology, applied sciences and engineering. The tools of fractional calculus are found to be of great support in developing a more realistic mathematical modeling of the applied problems in terms of fractional differential equations. Fractional-order models are regarded as better than the classical ones (based on differential equations) as fractional derivatives can take care of the hereditary properties of materials and processes involved in the problem at hand. For details and explanations. In particular, a great interest has been shown by many authors in the subject of fractional-order boundary value problems (BVPs), and a variety of results for BVPs equipped with different kinds of boundary conditions have been obtained. Coupled systems of fractional-order differential equations constitute an interesting and important field of research in view of their applications in many real world problems such as anomalous diffusion, disease models, ecological models, synchronization of chaotic systems, etc.

We study a coupled system of nonlinear fractional differential equations in this paper:

\[
\begin{align*}
&cD^\alpha x(t) = f(t, x(t), y(t)), \quad 1 < \alpha \leq 2, \quad t \in [0, T] \\
&cD^\beta y(t) = g(t, x(t), y(t)), \quad 1 < \beta \leq 2, \quad t \in [0, T].
\end{align*}
\]

Supplemented with integral boundary conditions of the form:

\[
\begin{align*}
&\int_0^T x(s) \, ds = \rho_1 y(\zeta_1), \quad \int_0^T x'(s) \, ds = \rho_2 y'(\zeta_2) \\
&\int_0^T y(s) \, ds = \mu_1 x(\eta_1), \quad \int_0^T y'(s) \, ds = \mu_2 x'(\eta_2), \quad \eta_1, \eta_2, \zeta_1, \zeta_2 \in [0, T].
\end{align*}
\]

Where \( cD^k \), \( k = \alpha, \beta, \sigma, \gamma \) denote the Caputo fractional derivatives of order \( k \), \( k = \alpha, \beta, \sigma, \gamma \). Respectively, and \( f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R} \) are given continuous functions, and \( \rho_1, \rho_2, \mu_1, \mu_2 \) are real constants.

Here we emphasize that our problem is new in the sense of non-separated coupled boundary conditions introduced here. To the best of our knowledge, fractional-order coupled system has yet to be studied with the boundary conditions. In consequence, our findings of the present work will be a useful contribution to the existing literature on the topic. The existence and uniqueness results for the given problem are new, though they are proved by applying the well-known method based on Banach’s contraction principle and Leray-Schauder’s alternative. The rest of the contents of the paper is organized as follows. In the second part we recall some basic definitions of fractional calculus and present an auxiliary lemma, which plays a pivotal role in obtaining the main results presented in the last section. We also discuss an example for illustration of the existence-uniqueness result. The paper concludes with some interesting observations.
Let \( w, z \in C \left[ [0, 1], R \right] \) then the unique solution for the problem

\[
\begin{align*}
\left \{ \begin{array}{ll}
\mathcal{D}^\alpha x (t) = w (t), & t \in [0, T], \quad 1 < \alpha \leq 2, \\
\mathcal{D}^\beta y (t) = z (t), & t \in [0, T], \quad 1 < \beta \leq 2, \\
\int_{0}^{t} x (s) \, ds = \rho_{1} y (\xi_{1}), & \int_{0}^{t} x' (s) \, ds = \rho_{2} y' (\xi_{2}) \\
\int_{0}^{t} y (s) \, ds = \mu_{1} x (\eta_{1}), & \int_{0}^{t} y' (s) \, ds = \mu_{2} x' (\eta_{2})
\end{array} \right.
\end{align*}
\]

is

\[
x (t) = \frac{T_{1} \int_{0}^{t} \rho_{2} (1^{3}-1^{1}) (\xi_{2}) - (1^{0} w) (T) + \frac{\rho_{2}^{2}}{T^{2}} (1^{a}-1^{1}) (\eta_{2}) - \frac{\rho_{2}^{3}}{T^{3}} (1^{\beta} z) (T)}{1^2 - \rho_{1} \mu_{1}}
\]

and

\[
y (t) = \frac{T_{1} \int_{0}^{t} \rho_{2} (1^{3}-1^{1}) (\xi_{2}) - \frac{\rho_{2}^{3}}{T^{3}} (1^{a}-1^{1}) (\eta_{2}) - \frac{\rho_{2}^{2}}{T^{2}} (1^{b} z) (T)}{1^2 - \rho_{1} \mu_{1}}
\]

where

\[
\begin{align*}
\psi &= \frac{2 T^2 \rho_1 \xi_1 - T^3 \rho_2 + 2 \rho_1 \eta_1 \mu_1 \rho_2 - T^3 \rho_1}{2 (T^2 - \mu_2 \rho_2)}, \\
\lambda &= \frac{T (2 \mu_1 \eta_1 \rho_2 - T^3 + 2 \rho_1 \xi_1 \mu_1 - T \mu_1 \rho_2)}{2 (T^2 - \mu_2 \rho_2)}, \\
\Delta &= \frac{2 T^3 \mu_1 \eta_1 - T^3 \mu_2 + 2 \rho_1 \xi_1 \mu_1 \mu_2 - T^3 \mu_1}{2 (T^2 - \mu_2 \rho_2)}, \\
\delta &= \frac{T (2 \rho_1 \xi_1 \mu_2 - T^3 + 2 \rho_1 \eta_1 \mu_1 - T \mu_2 \rho_1)}{2 (T^2 - \mu_2 \rho_2)}
\end{align*}
\]

To define the solution for the problem (1), we state the following auxiliary lemma.

**Lemma 1.** Let \( w, z \in C \left[ [0, 1], R \right] \). Then the unique solution for the problem

\[
\begin{align*}
\left \{ \begin{array}{ll}
\mathcal{D}^\alpha x (t) = w (t), & t \in [0, T], \quad 1 < \alpha \leq 2, \\
\mathcal{D}^\beta y (t) = z (t), & t \in [0, T], \quad 1 < \beta \leq 2, \\
\int_{0}^{t} x (s) \, ds = \rho_{1} y (\xi_{1}), & \int_{0}^{t} x' (s) \, ds = \rho_{2} y' (\xi_{2}) \\
\int_{0}^{t} y (s) \, ds = \mu_{1} x (\eta_{1}), & \int_{0}^{t} y' (s) \, ds = \mu_{2} x' (\eta_{2})
\end{array} \right.
\end{align*}
\]

is

\[
x (t) = \frac{T_{1} \int_{0}^{t} \rho_{2} (1^{3}-1^{1}) (\xi_{2}) - (1^{0} w) (T) + \frac{\rho_{2}^{2}}{T^{2}} (1^{a}-1^{1}) (\eta_{2}) - \frac{\rho_{2}^{3}}{T^{3}} (1^{\beta} z) (T)}{1^2 - \rho_{1} \mu_{1}} + (1^{3} z) (t),
\]

and

\[
y (t) = \frac{T_{1} \int_{0}^{t} \rho_{2} (1^{3}-1^{1}) (\xi_{2}) - \frac{\rho_{2}^{3}}{T^{3}} (1^{a}-1^{1}) (\eta_{2}) - \frac{\rho_{2}^{2}}{T^{2}} (1^{b} z) (T)}{1^2 - \rho_{1} \mu_{1}} + (1^{b} z) (t),
\]

where

\[
\begin{align*}
\psi &= \frac{2 T^2 \rho_1 \xi_1 - T^3 \rho_2 + 2 \rho_1 \eta_1 \mu_1 \rho_2 - T^3 \rho_1}{2 (T^2 - \mu_2 \rho_2)}, \\
\lambda &= \frac{T (2 \mu_1 \eta_1 \rho_2 - T^3 + 2 \rho_1 \xi_1 \mu_1 - T \mu_1 \rho_2)}{2 (T^2 - \mu_2 \rho_2)}, \\
\Delta &= \frac{2 T^3 \mu_1 \eta_1 - T^3 \mu_2 + 2 \rho_1 \xi_1 \mu_1 \mu_2 - T^3 \mu_1}{2 (T^2 - \mu_2 \rho_2)}
\end{align*}
\]
\[ \delta = \left( \frac{T(2\rho_1\xi_1\mu_2 - T^3 + 2\rho_1\eta_1\mu_1 - T\mu_2\rho_1)}{2(T^2 - \mu_2\rho_2)} \right). \]

**Theorem 1.** Assume that
(i): \( f, g : [0, T] \times R \to R \) are jointly continuous functions;
(ii): there exist a constants \( l_f, l_g \in R_+ \), such that \( \forall x_1, x_2, y_1, y_2 \in R, \forall t \in [0, T] \), we have
\[ |f(t, x_1, x_2) - f(t, y_1, y_2)| \leq l_f (|x_1 - x_2| + |y_1 - y_2|), \]
\[ |g(t, x_1, x_2) - g(t, y_1, y_2)| \leq l_g (|x_1 - x_2| + |y_1 - y_2|). \]

If
\[ l_f (Q_1 + Q_3) + l_g (Q_2 + Q_4) < 1 \]
then the BVP (1) has a unique solution on \([0, T]\). Here
\[ Q_1 = \frac{T}{T^2 - \mu_2\rho_2} \left| \frac{T^\alpha + 1 + \beta\mu_2\rho_2}{\Gamma(\alpha + 1)} \right| + \frac{1}{T^2 - \mu_1\rho_1} \left| \frac{(\alpha + \beta)\mu_2\rho_2}{\Gamma(\alpha + 2)} \right|, \]
\[ Q_2 = \frac{T}{T^2 - \mu_2\rho_2} \left| \frac{\rho_2}{\Gamma(\beta + 1)} \right| + \frac{1}{T^2 - \mu_1\rho_1} \left| \frac{(\alpha + \beta)\mu_2\rho_2}{\Gamma(\alpha + 2)} \right|, \]
\[ Q_3 = \frac{T}{T^2 - \mu_2\rho_2} \left| \frac{\rho_1}{\Gamma(\beta + 1)} \right| + \frac{1}{T^2 - \mu_1\rho_1} \left| \frac{(\alpha + \beta)\mu_2\rho_2}{\Gamma(\alpha + 2)} \right|, \]
\[ Q_4 = \frac{T}{T^2 - \mu_2\rho_2} \left| \frac{\Delta T^{\alpha - 1} + \beta\mu_2\rho_2}{\Gamma(\alpha + 1)} \right| + \frac{1}{T^2 - \mu_1\rho_1} \left| \frac{(\alpha + \beta)\Delta T^{\alpha - 1}}{\Gamma(\alpha + 2)} \right|, \]
\[ \frac{\Delta T^{\beta - 1} + \beta\mu_2\rho_2}{\Gamma(\beta + 1)} + \frac{\Delta T^{\beta - 1} + \beta\mu_2\rho_2}{\Gamma(\beta + 2)} + \frac{\beta_1\rho_2}{\Gamma(\beta + 2)} \]

**Theorem 2.** Assume that
(i): \( f, g : [0, T] \times R \to R \) are jointly continuous functions;
(ii): there exist a constants \( \exists \theta_0, \lambda_0 > 0 \) and \( \lambda_1, \lambda_2, \theta_1, \theta_2 \geq 0 \) where \( \theta_0, \lambda_0, \lambda_1, \lambda_2, \theta_1, \theta_2 \) are real constant such that \( \forall x_i, y_i \in R, (i = 1, 2) \) we have
\[ |f(t, x_1, x_2)| \leq \theta_0 + \theta_1 |x_1| + \theta_2 |x_2|, \]
\[ |g(t, x_1, x_2)| \leq \lambda_0 + \lambda_1 |x_1| + \lambda_2 |x_2|. \]

If
\[ (Q_1 + Q_3) \theta_1 + (Q_2 + Q_4) \lambda_1 < 1, \]
\[ (Q_1 + Q_3) \theta_2 + (Q_2 + Q_4) \lambda_2 < 1 \]
then the BVP (1) has at least one solution on \([0, T]\).

**Keywords:** Coupled system, fractional differential equations, boundary conditions.

**AMS Subject Classification:** 34A08.

**References**

APPLICATION OF FUZZY LOGIC FOR RISK DETERMINATION OF TYPE 2 DIABETES DISEASE

NOVRUZ ALLAHVERDI¹, NESRIN ERTOSUN¹

¹Computer Engineering Department, KTO Karatay University, Konya, Turkey
 e-mail: novruz.allahverdi@katatay.edu.tr, nesrinertosun@gmail.com

Diabetes is a disease caused by insufficient production of insulin by the organ called pancreas. If the insulin secretion in the patient is absent or too small, then it is called diabetes type 1; if the amount of insulin or its effect is insufficient, it is called diabetes type 2.

Nowadays, the topic of diabetes type 2 is becoming increasingly prevalent, even at very young ages. Of course, this spreading has multiple causes. People with diabetes in their family, fat people, people living under stress, are more likely to be sick with diabetes. It is sufficient to several values in the patient’s blood analysis for the determination of diabetes type 2. However, determining the exact value of the risk of this disease is difficult for some reasons.

In this study, a fuzzy control system is presented to aid physicians in the diagnosis and to identify a risk value of type 2 diabetes.

Fuzzy systems have become increasingly used in medicine. A number of studies have been conducted on the diagnosis of both types of diabetes mellitus and other according procedures [1,2,5–7].

The diagnosis of diabetes mellitus is made by evaluating the symptoms of the disease and blood glucose measurements together. The “fasting blood sugar” measured after at least 8 hours of fasting should be below 100 mg/dl. Diabetes is diagnosed if any of the following conditions exist: (1) The blood sugar level measured at any time of the day exceeds 200 mg/dl and diabetic complaints; (2) The fasting blood sugar level is 126 mg/dl or more; (3) The blood sugar level for the 2nd hour after the test for a sugar test using 75 g of glucose solution is 200 mg/dl or more; (4) The value of the component of glycosylated hemoglobin (HbA1c) exceeds 6.5 mg/dl [4].

The following factors can play a role in diabetes: (a) genetic factors; (b) being overweight and eating high caloric; (c) inactivity; (d) abnormal amount of glucose produced in the liver; (e) metabolic syndrome; (f) delivering a child over 4 kilos; (g) high blood pressure and high stress; (h) the attack of the immune system to the beta cells with error; (i) exposure to certain harmful microorganisms or environment containing toxins.

In order to be able to diagnose diabetes, you need to look at the value of HbA1c, which gives information on the fasting blood sugar, the amount of sugar in the blood serum, the percentage of blood cells covered with glucose in the stomach.

To determine the risk of type 2 diabetes we designed a simple fuzzy expert system. The inputs of such a system will be the fasting and the satiety blood sugar levels, HbA1c and the weight of the patient and the output will be the percentage of risk of diabetes. All inputs and outputs are divided into 4 fuzzy sets for more precise examination of the results of the blood test of the person [3].

As an example (Table 1), a fuzzy membership ”Low” status of fasting blood sugar amount values, includes lower numerical values in the normal human fasting blood sugar value. That is, the range of 50 mg/dl to 90 mg/dl was defined as the low range. Secondly, it is a ”normal” fuzzy
set, in which the fasting blood sugars range that a normal person should have in the test result (from 70 mg/dl to 130 mg/dl). Thirdly, it is a “High” membership, in which the numerical range of blood test values (from 110 mg/dl to 170 mg/dl) was determined to be higher than the normal value of fasting blood sugar. The last fourth member is the fasting blood sugar entry, “Very High”, contains values that we can say critical (no more than 150 mg/dl to 200 mg/dl and over).

Table 1. Min and max value range of inputs and outputs of generated system [3].

<table>
<thead>
<tr>
<th>Inputs and Output</th>
<th>Low</th>
<th>Normal</th>
<th>High</th>
<th>Very High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fasting blood sugar (mg/dl)</td>
<td>50-90</td>
<td>70-130</td>
<td>110-170</td>
<td>150-200 and over</td>
</tr>
<tr>
<td>Satiety blood sugar (mg/dl)</td>
<td>80-120</td>
<td>100-160</td>
<td>140-200</td>
<td>180-230</td>
</tr>
<tr>
<td>HBA1C (%)</td>
<td>1-4,5</td>
<td>2,5-8</td>
<td>6-11,5</td>
<td>9,5-15</td>
</tr>
<tr>
<td>Weight (kg)</td>
<td>35-70</td>
<td>50-105</td>
<td>140-200</td>
<td>180-230</td>
</tr>
<tr>
<td>Percentage of Risk of Diabetes (%)</td>
<td>1-30</td>
<td>26-59</td>
<td>55-88</td>
<td>85-100</td>
</tr>
</tbody>
</table>

The following 1 to 4 formulas were obtained for the membership functions of the fasting blood glucose levels “Low”, “Normal”, “High” and “Very High” [3]:

Figure 1 graphically depicts the Fasting Blood Glucose entry organized according to these formulas.

Figure 1. Graphical Presentation of Fuzzy Fasting Blood Sugar (One of the Inputs).

The output value of this system (Diabetes Risk) has also been specified as 4 membership functions Low, Normal, High and Very High (Fig.2).

\[
\mu_{Low}(x) = \begin{cases} 
1, & \text{if } 50 \leq x \leq 70 \\
(90 - x)/(90 - 70), & \text{if } 70 < x \leq 90 
\end{cases} 
\]

(1)

\[
\mu_{Normal}(x) = \begin{cases} 
1, & \text{if } 90 \leq x < 110 \\
(130 - x)/(130 - 110), & \text{if } 110 < x \leq 130 
\end{cases} 
\]

(2)

\[
\mu_{High}(x) = \begin{cases} 
1, & \text{if } 130 \leq x \leq 150 \\
(170 - x)/(170 - 150), & \text{if } 150 < x \leq 170 
\end{cases} 
\]

(3)

\[
\mu_{VeryHigh}(x) = \begin{cases} 
(x - 150)/(170 - 150), & \text{if } 150 \leq x \leq 170 \\
1, & \text{if } x \geq 170 
\end{cases} 
\]

(4)
The numbers of the inputs membership function is 4 (Fasting Blood Sugar, Satiety Blood Sugar, HbA1c, and Weight), and each of them has 4 fuzzy values (Low, Normal, High and Very High), so we will have \(4 \times 4 \times 4 \times 4 = 256\) fuzzy rules must be written to evaluate all conditions of the input variables.

The developed fuzzy rules which will fire by the obtained membership degrees were arranged by the aid a doctor. To determinate Diabetes Risk, we have used Matlab fuzzy toolbox. Let us see the next examples [3].

Example 1: Fasting Blood Sugar = 125 mg/dl, saturation blood sugar = 155 mg/dl, HbA1c = 8% and Weight = 90 kg. In this case, when the system will start, five rules will be put into action, and the system will find the diabetes ratio as 71.3%. Here the Mean of a Maximum method is chosen as the defuzzifier. If we use the Centroid method as a defuzzifier, the risk of diabetes will be 57%.

Example 2: Fasting Blood Sugar=200 mg/dl; Satiety Blood Sugar=230 mg/dl, HbA1c=%7, and Weight=85 kg. For this patient, five rules will fire and the system finds Diabetes Risk as % 92.7.

So, the projected fuzzy expert system helps doctors when they take a decision about the severity of the disease type 2 diabetes.

**Keywords:** Diabetes type 2, fuzzy expert system, blood glucose, diabetes risk.

**AMS Subject Classification:** 93C42.

**REFERENCES**


Singularly perturbed differential equations are typically characterized by a small parameter $\varepsilon$ multiplying some or all of the highest order terms in the differential equation. In general, the solutions of such equations exhibit multiscale phenomena. History of numerical methods for singularly perturbed problems begins about 40 years ago, in 1968. The initial era of singularly perturbed problems is 1968-1984. Singular perturbations were first described by Prandtl in a seven-page report presented at the Third International Congress of Mathematicians in Heidelberg in 1904. However, the term ”singular perturbations” was first used by Friedrichs and Wasow in a paper presented at a seminar on nonlinear vibrations at New York University. Singularly perturbed differential equations refer to the study of a class of differential equations containing an asymptotically small parameter where the character of the limiting solution is completely different from the solutions obtained at finite values of the parameter. For instance, in two dimensions the Navier-Stokes equations comprise the system of four non-linear partial differential equations for the conservation of mass, momentum and energy reduced from being non-linear parabolic equations to the system of four first-order non-linear hyperbolic differential equations known as Euler equations. Thus the principal aim to study the singularly perturbed parabolic problems is to explore the ”no-mans land” between the parabolic and hyperbolic realms. The study of many theoretical and applied problems in science and technology leads to boundary value problems for singularly perturbed differential equations that have a multiscale character. However, most of the problems cannot be completely solved by analytic techniques. Consequently, numerical simulations are of fundamental importance in gaining some useful insights on the solutions of the singularly perturbed differential equations. These singularly perturbed problems arise in the modeling of various modern complicated processes, such as fluid flow at high Reynolds numbers, water quality problems in rivers: networks, drift diffusion equation of semiconductor device modeling, financial modeling of option pricing, turbulence model, simulation of oil extraction from under-ground reservoirs, atmospheric pollution, groundwater transport and chemical reactor theory. In the modeling of these processes, characterized by dominant convection and/or intensive reactions, one can observe boundary and interior layers whose width, depending on the perturbation parameters, can be arbitrarily small. On the other hand, the domain itself, where the problem in question is considered, can be extremely large, even unbounded, compared to the available computational resources. Standard numerical methods applied to such multiscale problems give seemingly large errors, which make these methods inapplicable for practical use. The solution of singular perturbation problems typically contains layers and this causes various complications in the numerical treatment of the equations. Prandtl originally introduced the term ”boundary layer”, but this term came into more general use following the work of Wasow.
Pearson was among the first to consider the finite-difference method to solve singular perturbation problems. Consider the simple singularly perturbed first order initial value problem \((P_{\varepsilon})\) on the interval \(\Omega = (0, X)\)

\[
\varepsilon u'_x + a(x)u_x = f(x), \quad x \in \Omega
\]

\(u_x(0)\) given, where, for all \(x \in \Omega, a(x) \geq \alpha\) and \(0 < \varepsilon \leq 1\).

We are interested in designing a numerical method which gives good approximations to the solution of \((P_{\varepsilon})\), regardless of the value of \(\varepsilon\) in the entire range \(0 < \varepsilon \leq 1\). To analyse such problems and their numerical solutions introduce some norms and semi-norms. For singular perturbation problems it is important to work in the maximum norm. We define the maximum norm of a differentiable function on a set \(S\).

**Definition 1.**

\[|\Phi|_S = \sup_{x \in S} |\Phi(x)|\]

and for any positive integer \(k\), we define the \(k\)-th order semi-norm of a differentiable function on a set \(S\).

**Definition 2.**

\[|\Phi|_{k,S} = \sup_{x \in S} |\Phi^{(k)}(x)|.\]

For \(k = 0\), the semi-norm becomes the maximum norm. For some \(X > 0\), let

\[\Omega = (0, X)\quad \text{and}\quad \Omega = [0, X].\]

For convenience we introduce the differential operator

\[L_\varepsilon = \varepsilon \frac{d}{dx} + a(x), \quad a(x) > \alpha > 0, \quad \text{for all} \quad x \in \Omega.\]

The operator \(L_\varepsilon\) satisfies the following maximum principle.

**Lemma 1.** Let \(\psi(x)\) be any function in the domain of \(L_\varepsilon\) such that \(\psi(0) \geq 0\). Then \(L_\varepsilon \psi(x) \geq 0\) for all \(x \in \Omega\) implies that \(\psi(x) \geq 0\) for all \(x \in \Omega\).

**Lemma 2.** Let \(\psi(x)\) be any function in the domain of \(L_\varepsilon\). Then

\[|\psi(x)| \leq \max\{|\psi(0)|, \frac{1}{\alpha}|L_\varepsilon \psi|\} \quad x \in \Omega.\]

In order to discuss numerical solutions we need to discretise the domain \(\Omega = (0, X)\). The simplest discretisation is a uniform mesh having \(N\) sub-intervals of equal length \(h\), which is determined by a set of \(N + 1\) equally spaced points \(\Omega^N_h = \{x_i\}_{i=0}^N\). Here, \(x_0 = 0, x_N = X\) and for any \(i, 1 \leq i \leq N, h = x_i - x_{i-1}\).

The using of classical difference methods for solving singularly perturbed problems may give rise to difficulties when the singular perturbation parameter \(\varepsilon\) is small. Therefore, it is important to develop suitable numerical methods to these problems. In general: for linear problems it is adequate to use exponentially fitted schemes. Uniform meshes are adequate for fitted operator methods, but piecewise-uniform meshes are used for fitted mesh methods. This approach is now widely used for numerical solutions of differential equations with step, continuous solutions. Especially this type of schemes are being used in nonlinear singular perturbation problems. A Shishkin mesh is a piecewise uniform mesh. What distinguishes a Shishkin mesh from any other piecewise uniform mesh is the choice of the so-called transition parameter(s), which are the point(s) at which the mesh size changed abruptly. Piecewise-uniform meshes are the simplest kind of non-uniform mesh and they are constructed as follows. The interval \((0, X)\) is divided into two pieces \((0, \sigma)\) and \((\sigma, X)\). \(N\) is chosen to be an even number and each piece is discretised by a uniform mesh with \(\frac{N}{2}\) sub-intervals. If the point \(\sigma = \frac{1}{2}\), it is clear that the complete mesh \(\Omega^N_h = \{x_i\}_{i=0}^N\) on \(X\) is a uniform mesh, but if \(\sigma \neq \frac{1}{2}\), say \(\sigma < \frac{1}{2}\), then the mesh in \((0, \sigma)\) is finer than the mesh in \((\sigma, X)\) and the point \(\sigma\) is called the transition point. Here \(x_0 = 0, \ x_N = \sigma, \ x_N = X\). In the fine mesh the mesh spacing is \(x_i - x_{i-1} = \frac{2\sigma}{N}\), while in the coarse mesh it is \(\frac{2(X-\sigma)}{N}\). In this brief survey we reviewed methods developed by numerous researchers after 2005.
The considered problems include linear, non-linear and reaction-diffusion types. The numerical techniques reviewed in this survey include finite-difference and computational methods for initial and boundary value techniques.

**Keywords:** Singular perturbation, boundary layers, layer adapted meshes, uniform convergence.

**AMS Subject Classification:** 65L11, 65L12, 65L20, 65L70.

**References**


ON THE BOUNDARIES OF CHANGING PARAMETERS IN THE
MATHEMATICAL MODELING OF THE DYNAMIC SYSTEM
OF THE FIGURE OF THE EARTH

G.T. ARAZOV¹, T.H. ALIYEVA²

¹Institute of Applied Mathematics, Baku State University
²Institute of Physical Problems, Baku State University
e-mail: arazov_h@yahoo.com

Statistical regularities refer only to estimates that are found from observations. We denote by \( O(t₁), O(t₂), ..., O(tₙ) \) — numbers corresponding to statistical data, found from observations at times \( t₁, t₂, ..., tₙ \); \( C(t₁), C(t₂), ..., C(tₙ) \) — the numbers found from the calculations, from the formulas of mathematical modeling, for the same moments of time. The values \( O(tₙ) \) and \( C(tₙ) \) satisfy inequation:

\[
\sum_{n=1}^{∞} R(tₙ) \leq O(tₙ) - C(tₙ) \leq \sum_{n=1}^{k} F(tₙ), \quad (k = 1, 2, ...),
\]

where \( \sum_{n=1}^{∞} R(tₙ) \) — the sum of infinitesimal perturbations (actions) that are elusive by direct measurements. They have a hidden effect on the values of observations and are regulators of sets of natural phenomena. \( \sum_{n=1}^{k} F(tₙ) \) — the sum of the forces acting, under the influence of which certain processes are happening that are captured by mathematical modeling. The statistical data and patterns corresponding to them, over time, undergo various changes that correspond to disturbing actions. They are associated with the prerequisites of mathematical modeling, and can be determined approximately. This is due to the elusive sum of infinitesimal perturbations and resonance phenomena between the elements of the dynamic system that continuously change with time, as well as the configuration of the elements of the system. In all other cases, dynamic systems are under enormous pressure of numerous nonlinear fields surrounding them. In addition, they are influenced by various resonant phenomena. The evolution of these systems occurs under the influence of the sums of an infinite set of both small and large forces. They are able to change the stability of the system to instability, and vice versa. Thus, the evolution of dynamic systems as well as the connections between its elements depends on perturbations and resonance phenomena, both internal and external objects of the system. The speed of evolution of distances between the Earth’s continents, during 1972-1986, according to [1, p.190], can be rewritten as Table 1.

Table 1. Speed of change of distances between Earth continents.

<table>
<thead>
<tr>
<th>Distance (chord length) between</th>
<th>Increases with speed ( v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Europe and North America</td>
<td>( 1 \leq v \leq 2 \text{sm/year} )</td>
</tr>
<tr>
<td>North America and Hawaii</td>
<td>( 3,1 \leq v \leq 4,2 \text{sm/year} )</td>
</tr>
<tr>
<td>Hawaii and South America</td>
<td>( 2 \leq v \leq 8 \text{sm/year} )</td>
</tr>
<tr>
<td>South America and Australia</td>
<td>( 3 \leq v \leq 9 \text{sm/year} )</td>
</tr>
<tr>
<td>Australia and Hawaii</td>
<td>( 6 \leq v \leq 8 \text{sm/year} )</td>
</tr>
</tbody>
</table>
From Table 1 it follows that the size, shape, and internal structure of the Earth, continuously changing with time and they go the way of the evolution of statistical patterns. Thus, the statistical patterns of the evolution of the elements of the outer form and the internal structure of the Earth are combined in the form of motions of continents with different velocities: $1 \leq v \leq 9\; sm/\text{year}$. Changes in these boundaries, both in the past and in the future, can only be investigated by mathematical modeling. And this can be realized with the help of statistical estimates found from observations. All observed phenomena of nature reflect a set of specific statistical data and statistical regularities. The results of qualitative analysis of mathematical modeling depend on the time interval covered by the estimates of observations. In this case, one must not forget about the perturbations and evolution of various combinations of the elements of the system. Estimations of the results of observations constitute only statistical regularities of the observed processes and phenomena. All the rest belongs to the model problem, which is formed on the basis of various simplifications and assumptions. To control the correctness of the model problem (mathematical modeling), it must satisfy known statistical data. Collected by millennia of observation data, as well as the current state of the Earth allow for a different modeling of the evolution of both the external form and its internal structure. However, none of these models can cover completely the evolution of Earth processes and phenomena over geological time intervals. According to [1], the velocity of the secular change in the mean radius of the Earth $R$ of the expanding model of the Earth can be represented by the formula:

$$
\frac{dR}{dt} = (2,8 \pm 0,8)\; sm/\text{year}
$$

i.e. varies between boundaries:

$$
2\; sm/\text{year} \leq \frac{dR}{dt} \leq 3,6\; sm/\text{year}.
$$

The validity of these estimates is shown in three different ways. Laser measurements carried out using both 1) the surface of the Moon, and 2) artificial Earth satellites such as Lageos; 3) measurements of long-baseline interferometers. In the works of Yu.B. Barkin [2; 3], in three different ways it is shown that the rate of secular changes in the mean radius of a compressible Earth model can be determined by the formula:

$$
\frac{dR}{dt} = -(0,273 \pm 0,031)\; mm/\text{year},
$$

in other words, varies between boundaries:

$$
-0,304\; mm/\text{year} \leq \frac{dR}{dt} \leq -0,242\; mm/\text{year}.
$$

The mathematical model represents a certain number of formulas. They express the relationship between the measurable magnitude of the problem and the parameters of the proposed model. The accuracy of these formulas shows how much this model corresponds to reality. Relations between the parameters of expanding and contracting geodynamic models with measurable parameters of the Earth can be represented by the formulas:

$$
I_n = 0,25 \left(\frac{c^n}{R^n}\right) \cdot \left[(1 + i\delta)(\delta + i)^n + (1 - i\delta)(\delta - i)^n\right], \quad (n \geq 2, \; i = \sqrt{-1}),
$$

where $I_n$— parameters characterizing the structure of the Earth, $c$ and $\delta$— parameters of the Jacobi dynamic system. The radius of the vectors and acceleration of gravity at various parallels of the Earth’s surface, i.e. $[r_n(\varphi), g_n(\varphi)]$ are determined from observations. The equation of the level surface corresponding to the secular variation of the average equatorial radius of the Earth (4) can be expressed by the formula:

$$
r_n = \left[R_0 - 2,73 \cdot 10^{-5}(t - t_0)\right] \cdot \left[1 - 1,5(I_2)_n \sin \varphi + 2,5(I_3)_n \cdot \sin^3 \varphi -
-0,5 [q - 3(I_2)_n + 7,5(I_3)_n - 4,5(I_2)_n^2 + q^2] \sin^2 \varphi +
+0,75 [q^2 - 3,5(\gamma_2) - 6(I_2)_n^2] \sin^3 \varphi + ...ight]
$$
where \( q = \omega^2 R_0^2 / (\ell m) = 0.00346142 \). Secular changes in the acceleration of gravity \( g_n \), depending on geographical latitude on a surface \( r_n \),

\[
g_n = \frac{f m}{R_0^2} \left[ 1 - \frac{2.23\times10^{-3}}{R_0^2} \cdot (t - t_0) \right]^{-2} \cdot \{ q1, 5(I_2)_n + 1, 875(I_3)_n - 3(I_3)_n \sin \varphi \}
\]

+ \[ 2q(1 + q^2) + 1, 5(I_2)_n(1 - 5q) + 9(I_2)_n^2 - 11, 25(I_3)_n^2 \sin^2 \varphi \]

+ \[ 5(I_3)_n \sin^3 \varphi \left[ 1 - 2, 75q^2 + (7, 5q(I_2)_n + 2, 25(I_2)_n^2 + 16, 625(I_3)_n) \sin^4 \varphi \right] \quad \ldots \]

Similarly \( r_n \) and formulas \( g_n = g_n(\varphi, t) \) can easily be obtained for the expanding geodynamic, mathematical model of the Earth. Using the statistical data given in [4], from (7) we obtain:

\[
(r_n' - r_n') (O - C) \leq 1 \text{ m}; (r_0 - r_0') (O - C) \leq 13 \text{ m}; (r_0 - r_0'') (O - C) \leq 15 \text{ m}, \tag{9}
\]

where \( r_n' = r_0(90^\circ) \) distance from the center of mass of the Earth to its north pole, \( (\varphi = 90^\circ); \)
\( r_n'' = r_n(-90^\circ) \) distance from the center of mass of the Earth to its south pole, \( (\varphi = -90^\circ); \)
\( r_0 = r_n(0^\circ) \) average radius of the equator of the Earth, \( (\varphi = 0^\circ) \). In this case, we used as the observed values:

\[
O(r_n' - r_n'') = 32 \text{ m}; \quad O(r_0' - r_n') = 21373 \text{ m}; \quad O(r_0 - r_0'') = 21466 \text{ m}. \tag{10}
\]

The compression of the middle, northern and southern hemispheres of the level surface (9) is equal to:

\[
\alpha = 1 - \frac{r_n' - r_n''}{2r_0} = \frac{1}{2} 298.12; \quad \alpha' = \frac{r_0 - r_n'}{r} = \frac{1}{2} 298.42; \quad \alpha'' = \frac{r_0 - r_n''}{r_0} = \frac{1}{2} 297.96. \tag{11}
\]

In addition, from the expression (10) we find:

\[
-5, 186 \leq g_n' - g_0 \leq 5, 230 \text{ gal}; \quad -5, 180 \leq g_n'' - g_0 \leq 5, 230 \text{ gal}, \tag{12}
\]

where \( g_n' = g_n(90^\circ); \quad g_n'' = g_n(-90^\circ) \) and \( g_0 = g_n(0^\circ) \) acceleration of gravity at the northern, southern poles equator of the Earth, respectively. The observed state of the Earth is the result of the total effects and evolution of numerous actions (currents). Therefore, the accuracy of observations, the time of observations, and also the duration of time intervals of observations play an important role in the dynamic processes of the formation and evolution of the Earth. All these factors are included in the statistical estimates of observations. At the same time, in dynamic models, many actions and consequences can not be taken into account i.e. their sum is less than the values of statistical estimates that represent the errors of observations.

**Keywords:** Dynamic systems, mathematical modeling, the figure of the Earth, evolution, stability.

**AMS Subject Classification:** 93M50, 97M10, 97M50.

**References**


DISCRETE OPTIMAL CONTROL WITH CLOSED LOOP POLES IN A CIRCULAR REGIONS IN FREQUENCY DOMAIN

CENGİZ C. ARCASOY¹, HÜLYA EROĞLU²

¹Department of Electrical and Electronics Engineering, Toros University, Mersin, Turkey
e-mail: arcasoy@hotmail.com

1. INTRODUCTION

This paper is concerned with the problem of multi-input, infinite time, linear time invariant quadratic cost discrete-time optimal control system with closed-loop poles in a circular regions centered at \( \beta \) with \( \alpha \) radius in \( z \)-plane. Depending on relations with \( \alpha \) and \( \beta \) four circular regions \( D_1, D_2, D_3, \) and \( D_4 \) are selected as shown in Figures 1-4. Using the frequency response algorithm [1], closed-loop poles of the system will be allocated in prescribed regions without solving the matrix Riccati equation. The new weighting matrices \( Q_1, R_1, \) and \( S_1 \) will be obtained by using the work of [3], for assignment of closed-loop poles in a mentioned regions.

2. PROBLEM STATEMENT

The method is based on spectral factorization of the performance spectrum matrix. The solution of the discrete optimal control problem described by its state equations will lead the discrete matrix Riccati equation. However, the frequency response form of the discrete Riccati equation can be written as [1]:

\[
F^T(z^{-1})(R + B^TPB)F(z) = R + G^T(z^{-1})QG(z) + S^T(zI - A)^{-1}B + B^T(z^{-1}I - A^T)^{-1}S = \psi(z) = \Delta^T(z^{-1})\Delta(z),
\]

where \( F(z)\psi \) is the discrete return difference matrix and \( G(z) \) is the discrete system transfer function matrix with:

\[
F(z) = I + K(zI - A)^{-1}B, \quad G(z) = C(zI - A)^{-1}B.
\]

The frequency-response algorithm for the determination of continuous-time optimal gain matrix \( K \) can be obtained from work of [2]. Once a continuous-time optimal gain matrix is obtained, the weighting matrix \( Q \) could be obtained by the frequency response method given in [2]. Then, using the work of [3], the discrete-time weighting matrices can be calculated easily. The design procedure, 7-step algorithm are given and examples are added to illustrate algorithm in the paper. The MATLAB code is used for calculations.
In the following figures we will consider regional pole assignment with conditions in complex z-plane as:

Figure 1. Region $D_1$: $\alpha < 1$, $\beta = 0$

Figure 2. $D_2$: $\alpha, \beta > 0$, $|\alpha| + |\beta| \leq 1$

Figure 3. Region $D_3$: $\alpha, \beta > 0$, $|\alpha| = |\beta| \leq 0.5$
3. Conclusion

As a conclusion, straightforward new algorithm to determine optimal weighting matrices and optimal gain matrix that place the closed-loop poles inside circular regions are presented in entirely frequency domain. The results are obtained by spectral factorization and therefore, without solving the discrete matrix Riccati equation. It is believed that the presented method is more direct than the algebraic matrix Riccati equation solution since the gain matrix can be evaluated directly in frequency domain without any iteration.

Keywords: Inverse optimal control, spectral factorization, pole placement.

AMS Subject Classification: 93C80, 93C55.

References


ABOUT GENERALIZED SOLUTIONS OF BASIC BOUNDARY VALUE PROBLEMS FOR THE SECOND ORDER ELLIPTIC EQUATION IN UNBOUNDED DOMAINS

T.B. ASADOV

1Baku State University, Baku, Azerbaijan
e-mail: tofig-as@mail.ru

Abstract. In the paper is considered the problems of the uniqueness and removable singularities solutions in unbounded domains of second-order linear elliptic equations of the basic boundary value problems for second-order elliptic equation in unbounded domains. The basic boundary value problems are formulated. The problems of the uniqueness and removable singularities of solutions are formulated as a theorem.

Keywords: Generalized solution, unbounded domain, elliptic equation, classical boundary value problems, removable singularities of the solution.

AMS Subject Classification: 35J15, 35J25.

1. Introduction.

The properties of generalized solutions of various boundary-value problems for second-order elliptic equations in the unbounded domain have been investigated quite extensively. The questions of the correctness of the formulation of boundary value problems are studied in different domains and classes of functions. The estimates of the solution in a wide class of unbounded domains are obtained and the accuracy of these estimates is established, the questions of the uniqueness and removable singularities of solutions of the basic boundary value problems are studied. In particular, in the work [7] an a priori estimate of the generalized solution of the mixed problem for a second-order linear elliptic equation in the theory of elasticity has been obtained. In this case, the domains with finite number of branches are considered, which sufficiently arbitrarily going to infinity. The boundary of the domain is divided into three parts, on which the boundary conditions of the first, second and third types are placed, respectively. In [4, 5] the solutions of exterior boundary value problems have been considered and corresponding theorem of the uniqueness of solutions has been established.

The uniqueness and the removable singularities of solutions of the basic boundary value problems are considered for the nonselfadjoint second-order elliptic equation in unbounded domains. These questions for the second-order elliptic equation in some other classes were considered in [1–3, 6–9].
2. Problem statement and main results

Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain. Consider the equation

$$
(a^{ij}(x)u_{x_i})_{x_i} + (b^i(x)u)_{x_i} + c(x)u(x) = f(x), x \in \Omega,
$$

where $\alpha_1 |\xi|^2 \leq a^{ij}(x)\xi_i \xi_j \leq \alpha_2 |\xi|^2$, $\alpha_1, \alpha_2 = \text{const} > 0$, $a^{ij}(x) = a^{ji}(x)$.

The summation over repeated indices from 1 to $n$ is assumed. The coefficients $a^{ij}(x)$ are bounded measurable functions in the domain $\Omega \subset \mathbb{R}^n$, $f \in L^2_{\text{loc}}(\Omega)$.

The following basic classical boundary-value problems for equation (1) are considered:

$$
u = 0 \partial \Omega,
$$

$$
\sigma (u) \equiv a^{ij}(x)u_{x_j}v_i = 0 \quad \text{on} \quad \partial \Omega,
$$

$$
\sigma (u) = 0 \quad \text{on} \quad \Gamma_1, \quad \sigma (u) = 0 \quad \text{on} \quad \Gamma_2, \quad \text{where} \quad \Gamma_1 \bigcup \Gamma_2 = \partial \Omega,
$$

$$
\sigma (u) + au = 0 \quad \text{on} \quad \partial \Omega. \quad \text{We assume that the function} \quad a(x) \geq 0 \quad \text{and}
$$

$$
a > 0 \quad \text{on} \quad \text{a set} \quad M \quad \text{positive measure}.
$$

Here $\nu = (\nu_1, ..., \nu_n)$ is the unit vector of the exterior normal to $\partial \Omega$, $\text{mes} \Gamma_1 \neq 0$, $\text{mes} \Gamma_2 \neq 0$.

Let $\omega$— the bounded domain in $\mathbb{R}^n$. We denote by $H^1(\omega, \Gamma)$ the class of functions obtained by completion of infinitely differentiable in $\overline{\omega}$ functions, equal to zero on $\Gamma$, in the norm

$$
\|u\|^2 = \int_{\omega} u \cdot u dx + \int_{\omega} u_{x_j} \cdot u_{x_j} dx.
$$

If $\Gamma = \emptyset$, then we denote $H^1(\omega, \Gamma)$ by $H^1(\omega)$. By a generalized solution of the problem (2) (respectively, problems (3) and (4)) we mean a function $v$ from the class $H^1(\omega_N, \partial \Omega \cap \partial \omega_N)$ (respectively from the class $H^1(\omega_N, H^1(\omega_N, \Gamma \cap \partial \omega_N))$) for any $N \geq N_0 = \text{const} > 0$, satisfying the integral identity

$$
- \int_{\omega_N} (a^{ij}(x)u_{x_j} \cdot v_i + b^i(x)u \cdot v_x - c(x)u \cdot v) dx = \int_{\omega_N} f \cdot v dx
$$

for any domain $\omega_N = \Omega \bigcap \{x : |x| < N\}$ and for any function $v \in H^1(\omega_N, \partial \omega_N)$ (respectively $v \in H^1(\omega_N, \partial \omega_N \cap \Omega)$, $v \in H^1(\omega_N, \partial \omega_N \cap \Omega) \bigcup (\partial \omega_N \cap \Gamma_1)$).

By a generalized solution of the problem (5) we mean a function $u$ from a class $H^1(\omega_N)$ satisfying the integral identity

$$
- \int_{\omega_N} (a^{ij}(x)u_{x_j} \cdot v_i + b^i(x)u \cdot v_x - c(x)u \cdot v) dx - \int_{\partial \omega_N \cap \partial \Omega} auv ds + \int_{\partial \omega_N \cap \partial \Omega} b^i(x)uv ds = \int_{\omega_N} f \cdot v dx,
$$

for any function $u$ from a class $H^1(\omega_N, \partial \omega_N \cap \Omega)$.

**Theorem 1.** Let the coefficients $a^{ij}(x)$ be bounded measurable functions, $b^i(x)$ have continuous first-order derivatives with respect to $x_i$, $b^i(x) \leq 0$ and $\max |x|^{1+\delta} |b^i| \leq M$, $c(x) \leq 0$.

Then the generalized solution of the problem (2), (4), (5) for equation (1) by $n = 2$ is unique, and the generalized solution of the problem (3) by $c(x) = 0$ is unique if

$$
|u(x)| \leq C |\ln |x||^{1-\varepsilon}
$$
for sufficiently large $|x|; C, \varepsilon = \text{const} > 0, \varepsilon > \frac{1}{2}$.

Now we formulate a theorem on removable singularities of solutions for equation (1).

**Theorem 2.** Let the function $u \in H^1(\omega^\delta)$ satisfies the integral identity

$$-\int_{\omega^\delta} (a^{ij}(x)u_{x_j} \cdot v_{x_i} + b^i(x)u \cdot v_{x_i} - c(x)u \cdot v) \, dx = \int_{\omega^\delta} f \cdot v \, dx,$$

for any $\delta, 0 \leq \delta \leq \frac{1}{2}$ and any function $v \in H^1(\omega^\delta, \partial \omega^\delta)$, where $\omega^\delta = \{ x : \delta < |x| < 1 \}$. Let

$$\max_{\omega^\delta} |b^i| \leq M, \frac{1}{2} b^i_{E_i}(E) + c(x) \leq 0, \|f\|_{L^2(\omega)} \leq c.$$

Suppose that

$$f \in L^2(\omega), \text{ for } n = 2 |x| < \frac{1}{2}, |u(x)| \leq c_1 |\ln |x||^{1-\varepsilon},$$

where $c_1, \varepsilon = \text{const} > 0, \varepsilon > \frac{1}{2}$. Then the singularity $u(x)$ at the point $E = 0$ is removable, i.e. $u(E) \in H^1(\omega)$ satisfies the integral identity

$$-\int_{\omega} (a^{ij}(x)u_{x_j} \cdot v_{x_i} + b^i(x)u \cdot v_{x_i} - c(x)u \cdot v) \, dx = \int_{\omega} f \cdot v \, dx$$

for any $v \in H^1(\omega, \partial \omega) \omega = \{ x : |x| < 1 \}$.

It should be noted that the smoothness of the coefficients of the elliptic equation plays an important role, which guarantees that its generalized solutions coincide with the classical ones and their belongings to the class of functions under consideration. For linear uniformly non-smooth elliptic equations, in particular, with discontinuous coefficients, the situation is more complicated, and the results on removable singularities of the solutions of such equations can differ substantially from the corresponding results for equations with smooth coefficients.

**References**


CONVERGENCE OF HP-STREAMLINE DIFFUSION AND NITSCHE’S SCHEMES FOR THE RELATIVISTIC VLASOV-MAXWELL SYSTEM

M. ASADZADEH¹, P. KOWALCZYK², C. STANDAR³

¹Department of Mathematics, Chalmers University of Technology and Göteborg University, Göteborg, Sweden
²Institute of Applied Mathematics and Mechanics, University of Warsaw, Poland
e-mail: mohammad@math.chalmers.se

1. Introduction

We study stability and convergence of $hp$-streamline diffusion (SD) finite element, and Nitsche’s schemes for the relativistic, time-dependent Vlasov-Maxwell (VM) system. We consider full physical domain with spatial variable $x \in \Omega_x \subset \mathbb{R}^3$ and 3-dimensional velocities $v \in \Omega_v \subset \mathbb{R}^3$. The objective is two-fold:

i) To derive globally optimal $a$ priori error bound of order $O(h^{s+1/2})$, for the SD approximation of the Vlasov-Maxwell system; where $h (= \max_K h_K)$ is the mesh parameter and $p (= \max_K p_K)$ is the spectral order. These estimates are based on the local version with $h_K$ being the diameter of the phase-space-time element $K$ and $p_K$ the spectral order (the degree of approximating finite element polynomial) for $K$. For the optimality of the $hp$ scheme for the Vlasov-Maxwell system, we need to assume that the exact solution is in the Sobolev space $H^{s+1}(\Omega)$.

ii) To improve the quality of Galerkin schemes for the Maxwell equations investigating combined effect of

a) The Nitsche’s symmetrization technique (cf [4], [11] and [1]) in the spatial scheme for a Galerkin method and
b) a time discretization, for a second order pde obtained through the combined Maxwell’s fields.

To construct Nitsche’s scheme, by a simple calculus of the field equations, we convert the Maxwell’s system to an elliptic type equation. Then, adequate symmetrization terms are introduced to the weak form. Combining the Nitsche’s method for the spatial discretization with a second order time scheme, we obtain optimal convergence of $O(h^2 + \tau^2)$, where $h$ is the spatial mesh size and $\tau$ is the time step. Here, as in the classical literature, the second order time scheme requires higher order regularity assumptions.

Numerical justification of these results, in lower dimensions, is the subject of a forthcoming work [10].

The motivation for this study In the SD method the weak form is modified by adding a multiple of the streaming part in the equation, to the test function. So we obtain a multiple of streaming terms in test and trial functions. This can be viewed as an extra diffusion in the streaming direction in the original equation. Hence, the name of the method (the streamline diffusion). Such an extra diffusion would improve both the stability and convergence properties of the underlying Galerkin scheme. It is well known that the standard Galerkin method has a weaker convergence property for the hyperbolic problems: $O(h^{s-1})$ versus $O(h^s)$ for the elliptic and parabolic problems with exact solution in the Sobolev space $H^s(\Omega)$. The SD method improves the convergence to $O(h^{s-1/2})$ and also, having an upwinding character, enhances the
2. Approaches to the Vlasov-Maxwell System

The Vlasov-Maxwell (VM) system describing the time evolution of collisionless plasma is formulated as

\[
\begin{align*}
\partial_t f + \mathbf{v} \cdot \nabla_x f + q(E + c^{-1}\mathbf{v} \times B) \cdot \nabla_v f &= 0, \\
\partial_t \rho &= E \nabla_x \times B - j, \\
\rho \nabla_x B &= -c \nabla_x E, \\
\nabla_x \cdot E &= \rho, \\
\nabla_x \cdot B &= 0
\end{align*}
\]

with properly assigned initial data \( f(0, x, v) = f^0(x, v) \geq 0, E(0, x) = E^0(x) \). Here \( f \) is the density, in phase space, time of particles with charge \( q \), mass \( m \) and velocity \( \mathbf{v} = (m^2 + c^{-2}|v|^2)^{-1/2}v \) (\( v \) is momentum). Further, \( c \) is the speed of light and the charge and current densities \( \rho \) and \( j \) are given by

\[
\rho(t, x) = 4\pi \int q f dv \quad \text{and} \quad j(t, x) = 4\pi \int q f \mathbf{v} dv.
\]

The Vlasov-Maxwell equations arise in several branches, e.g., continuum and plasma physics where the main assumption underlying the model is that collisions are rare and therefore negligible. In this setting the above system describes the motion of a collisionless plasma, e.g., a high-temperature, low-density, ionized gas. The mathematical concern in dealing with the Vlasov-Maxwell system is related to the nonlinear term: \((E + \mathbf{v} \times B) \cdot \nabla_v f\). Assuming divergence free field, this nonlinear term is written as \(\text{div}_v((E + \mathbf{v} \times B)f)\). Then a thorough mathematical analysis, with this nonlinearity, is given by I Perna Lions [6]. Numerical approaches for the VM system have been considered by several authors in different settings. To this work, the most relevant studies are given, e.g., by Gamba and co-workers [5] devoted to a discontinuous Galerkin approach, and Standar in [12] where the stability and a priori error estimates for the \( h \) version of SD method for VM are derived.

 Canonical forms. As a general framework we introduce similar convective forms for both Vlasov and Maxwell equations. The convection coefficients are nonlinear vector functions for the Vlasov equation and constant sparse multiple matrices in the Maxwell case. Despite physical diversities, these coefficients are exhibiting a common property: they both are divergence free, a property that substantially simplifies the analysis. Let \( E = (E_1, E_2, E_3)^T \), \( B = (B_1, B_2, B_3)^T \), \( j = (j_1, j_2, j_3)^T \), then the Maxwell’s equations can be written as:

\[
\begin{align*}
\partial_i E_1 &= \partial_2 B_3 - \partial_3 B_2 - j_1, \\
\partial_i E_2 &= \partial_3 B_1 - \partial_1 B_3 - j_2, \\
\partial_i E_3 &= \partial_1 B_2 - \partial_2 B_1 - j_3,
\end{align*}
\]

and for \( i = 1 \) or \( 2 \):

\[
\begin{align*}
\partial_i B_1 &= -\partial_2 E_3 + \partial_3 E_2, \\
\partial_i B_2 &= -\partial_3 E_1 + \partial_1 E_3, \\
\partial_i B_3 &= -\partial_1 E_2 + \partial_2 E_1.
\end{align*}
\]

where \( \partial_i \) denotes the derivative with respect to \( x_i \). We also introduce 6 \times 6 symmetric matrices \( M_k, k = 1, 2, 3 \) :

\[
\begin{cases}
(M_1)_{35} = (M_1)_{53} = -1, & \text{else } (M_1)_{ij} = 0, \\
(M_2)_{16} = (M_2)_{61} = -1, & \text{else } (M_2)_{ij} = 0, \\
(M_3)_{24} = (M_3)_{42} = -1, & \text{else } (M_3)_{ij} = 0.
\end{cases}
\]

To write the Maxwell equations in the matrix-vector form, we introduce the matrix-vector notation, viz. \( \mathbf{M} = (M_1, M_2, M_3) \), and set \( \mathbf{E} = (E_1, E_2, E_3), \ \mathbf{B} = (B_1, B_2, B_3), \ \mathbf{j} = (j_1, j_2, j_3), \ 0 =...\)
Further let \( W = (E, B)^T \) and use the notations for the initial values \( E^0, B^0, W^0 \). Then
\[
\partial_t W + M \cdot \nabla_x W = b, \quad \text{with} \quad W(0, x) = W^0(x).
\] (3)

Note that (3) is convective with, matrix-vector form, divergent free (\( \text{Div } M = 0 \)) convection coefficient.

The Vlasov part is of similar structure, but with nonlinear coefficient vectors depending on \( x \) and \( v \):
\[
\partial_t f + \hat{v} \cdot \nabla_x f + (E + \hat{v} \times B) \cdot \nabla_v f = 0, \quad \text{with} \quad f(0, x, v) = f^0(x, v) \geq 0,
\]
which can be rewritten in a compact form as
\[
\partial_t f + G(f) \cdot \nabla f = 0.
\] (4)

where \( \nabla f = (\nabla_x f, \nabla_v f) \) is the total gradient and \( G(f) = (\hat{v}, E + \hat{v} \times B) \) is divergence free:
\[
\text{div } G(f) = \sum_{i=1}^d \frac{\partial \hat{v}}{\partial x_i} + \sum_{i=d+1}^{2d} \frac{\partial (E + \hat{v} \times B)}{\partial v_{i-d}} = \nabla_v (\hat{v} \times B) = 0.
\] (5)

**Keywords:** Vlasov-Maxwell, Streamline-diffusion, convergence, stability.

**AMS Subject Classification:** 35L60, 65M60, 65N30, 82D10.

**References**


BIORTHOGONAL MULTIWAVELETS ON THE INTERVAL FOR SOLVING MULTI-DIMENSIONAL FRACTIONAL OPTIMAL CONTROL PROBLEMS WITH INEQUALITY CONSTRAINT

E. ASHPAZZADEH, M. LAKESTANI

Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran
e-mail: lakestani@tabrizu.ac.ir

1. Introduction

In this study, we consider the following multi-dimensional fractional optimal control problems. Find the q-vector control \( u(t) = [u_1(t), u_2(t), \ldots, u_q(t)]^T \), and the corresponding p-vector state \( f(t) = [f_1(t), f_2(t), \ldots, f_p(t)]^T \) which minimize(or maximize) the functional

\[
I(t, f(t), u(t)) = \int_0^1 g(t, f(t), u(t)) dt,
\]

subject to

\[
\begin{align*}
u(t) &= h(t, f(t), D^\alpha f(t)), \quad \alpha \geq 0, \quad n - 1 < \alpha \leq n, \quad 0 \leq t \leq 1, \\
R_i(t, f(t), D^\alpha f(t), u(t)) &\leq 0, \quad i = 1, 2, \ldots, v, \\
f^{(i)}(0) &= d_i, \quad i = 0, 1, \ldots, n - 1.
\end{align*}
\]

The vector function \( h \) and scalar functions \( g \) and \( R_j \) are generally nonlinear and assumed to be smooth with respect to \( x \) and \( u \). For the FOCP, we define the fractional derivative in the Caputo sense. This paper proposes a new numerical approach for solving multi-dimensional fractional optimal control problems (FOCPs) including state and control inequality constraints using new biorthogonal multiwavelets. The properties of biorthogonal multiwavelets are first given. The Riemann-Liouville fractional integral operator for biorthogonal multiwavelets is utilized to reduce the solution of optimal control problems to a nonlinear programming one, to which existing, well-developed algorithms may be applied. In order to save the memory requirement and computation time, a threshold procedure is applied to obtain algebraic equations. The method is computationally very attractive and gives very accurate results.

2. Biorthogonal multiwavelets for function approximation

The biorthogonal multiwavelets \( \psi^1(t) \) and \( \psi^2(t) \) are given by [1]

\[
\begin{align*}
\psi^1(t) &= \phi^1(2t) - \frac{1}{2}(\phi^1(2t + 1) + \phi^1(2t - 1)) - \frac{23}{12}(\phi^2(2t + 1) - \phi^2(2t - 1)), \\
\psi^2(t) &= \frac{37}{22}\phi^2(2t) + \frac{91}{88}(\phi^2(2t + 1) + \phi^2(2t - 1)) + \frac{1}{8}(\phi^1(2t + 1) - \phi^1(2t - 1)).
\end{align*}
\]

where \( \phi^1(t), \phi^2(t) \) are Hermite cubic splines.
3. Proposed method

By expanding the state rate $D^\alpha f(t)$ and $u(t)$ in terms of biorthogonal multiwavelet bases $\Psi_J(t)$ with unknown coefficients as

$$D^\alpha f(t) = F^T \Psi_J(t), \quad u(t) = U^T \Psi_J(t)$$

and utilizing the operational matrix of fractional Riemann-Liouville integration, we can approximate the dynamical system. For discretizing the functional $I$ in Eq. (1), we can express $g(f(t), u(t), t)$ in terms of biorthogonal multiwavelets as

$$g(t, f(t), u(t)) = \Gamma^T \Psi_J(t).$$

So

$$I = \int_0^1 \Gamma^T \Psi_J(t) dt = \Gamma^T P,$$

in which $P = \int_0^1 \Psi_J(t) dt$. Finally, we apply the inequality constraints in Eq. (3) with

$$R_i(t_k, f(t_k), u(t_k)) \leq 0, \quad i = 1, \ldots, v, \quad k = 1, \ldots, 2^J + 1.$$

In summary, the multi-dimensional fractional optimal control problem is discretized to the non-linear programming problem (NLP). Many well-developed NLP solvers can be used to solve this extremum problem.

4. Numerical results

Example 1. Find the optimal control $u(t)$ which [3]

Minimize $I = \int_0^1 -\ln 2f(t) dt,$

$$D^\alpha f(t) = \ln(2)(f(t) + u(t)),$$

$$|u(t)| \leq 1, \quad u(t) + f(t) \leq 2,$$

$$f(0) = 0.$$

The exact solution for $\alpha = 1$ is $I = -0.30682$. In Figure 1, the approximate state variable are plotted for different values of $\alpha$. 

Figure 1. Approximate solutions of $f(t)$ different values of $\alpha$ in Example 1.
Example 2. Consider the following nonlinear problem

\[
\text{Minimize } I = \int_0^1 \left[ f^2(t) - 2t^2 f(t) + u^2(t) - \frac{3\sqrt{\pi}}{4} e^{-t} u(t) + e^{-t+t^2} u(t) \right. \\
\left. + t^3 + \frac{9\pi}{64} e^{-2t} - \frac{3\sqrt{\pi}}{8} e^{-2t+t^2} + \frac{1}{4} e^{-2t+2t^2} + e^{2t} \right] dt,
\]

\[
D^{1.5} f(t) = e^f(t) u(t) + 2e^t u(t),
\]

\[f(0) = \dot{f}(0) = 0.\]

For this problem we have \(f(t) = \sqrt{t^3}\) and \(u(t) = \frac{1}{2} e^{-t} \left( e^{t^2} + \frac{3\sqrt{\pi}}{4} \right)\) with minimum value \(I = 3.19455\). In Figure 2., the exact and approximate state and control variables are plotted. Also Figure. 3, shows the plot of the matrix elements for \(J = 6\) after thresholding.

![Figure 2.](image-url)

Figure 2. Exact and approximate state and control variables for Example 2.

![Figure 3.](image-url)

Figure 3. Plots of sparse matrix after thresholding with \(\varepsilon = 10^{-3}\) for \(J = 6\), Example 2.

**Keywords:** Biorthogonal multiwavelets, Caputo derivative, Riemann-Liouville fractional integration, fractional optimal control problems, operational matrix.

**AMS Subject Classification:** 65Kxx, 65K99.

**REFERENCES**


TWO-BAND GINZBURG-LANDAU EQUATIONS: NUMERICAL SIMULATIONS*

I.N. ASKERZADE$^{1,2}$, R.T.ASKERBEYLI$^3$

$^1$Computer Engineering Department of Ankara University, Ankara, Turkey
$^2$Institute of Physics Azerbaijan National Academy of Science, Baku Azerbaijan
$^3$Karabuk University, Business Administration Dept, Karabuk, Turkey
e-mail: raskerbeyli@karabuk.edu.tr

1. Introduction

Numerical modeling of vortex lattice in external magnetic field in two-band superconductor using modified Ginzburg-Landau (GL) theory is conducted. Results of simulation experiments for a two-band superconducting films LiFeAs near the critical temperature in perpendicular magnetic field is presented and it was shown quasi-hexagonal character of vortex lattice.

2. Basic equations

Under external magnetic field without any restriction of generality, time-dependent equations in two-band Ginzburg-Landau theory can be written as [1-7]

\[
\begin{align*}
\Gamma_1 \frac{\partial \Psi_1}{\partial t} &= -\frac{\hbar^2}{4m_1} \left( \frac{d^2}{dx^2} - \frac{x^2}{l^2} \right) \Psi_1 + \alpha_1(T) \Psi_1 + \epsilon_1 \left( \frac{d}{dx} - \frac{x}{l} \right) \Psi_2 + \beta_1 \Psi_3^2 = 0, \\
\Gamma_2 \frac{\partial \Psi_2}{\partial t} &= -\frac{\hbar^2}{4m_2} \left( \frac{d^2}{dx^2} - \frac{x^2}{l^2} \right) \Psi_2 + \alpha_1(T) \Psi_2 + \epsilon_1 \left( \frac{d}{dx} - \frac{x}{l} \right) \Psi_1 + \beta_2 \Psi_3^2 = 0, \\
\sigma_n \left( \frac{\partial}{\partial r} - \nabla \phi \right) &= -\text{rot} A^r + \frac{2\pi}{\Phi_0} \left\{ \frac{\hbar^2}{4m_1} n_1(T) \left( \frac{d\phi_1}{dr} - \frac{2\pi A^r}{\Phi_0} \right) + \epsilon_1 \left( n_1(T)n_2(T) \right)^{0.5} \cos(\phi_1 - \phi_2) + \frac{\hbar^2}{4m_2} n_2(T) \left( \frac{d\phi_2}{dr} - \frac{2\pi A^r}{\Phi_0} \right) \right\},
\end{align*}
\]

with so-called natural boundary conditions

\[
\begin{align*}
\left\{ \frac{1}{4m_1} \left( \nabla - \frac{2\pi i A}{\Phi_0} \right) \Psi_1 + \epsilon_1 \left( \nabla - \frac{2\pi i A}{\Phi_0} \right) \Psi_2 \right\}^r &= 0, \quad n = 0. \tag{2a} \\
\left\{ \frac{1}{4m_2} \left( \nabla - \frac{2\pi i A}{\Phi_0} \right) \Psi_2 + \epsilon_1 \left( \nabla - \frac{2\pi i A}{\Phi_0} \right) \Psi_1 \right\}^r &= 0, \quad n = 0. \tag{2b} \\
\left( n^\rho \times A^r \right) \times n^\rho &= H_0^\rho \times n^\rho. \tag{2c}
\end{align*}
\]

First two conditions in (2) correspond to absence of supercurrent through boundary of two-band superconductor, third conditions correspond to the continuity of normal component of magnetic field to the boundary superconductor-vacuum.

In. Eqs. (1) $\Phi_{1,2}(r^\rho)$ phase of order parameters $\Psi_{1,2}(r^\rho) = |\Psi_{1,2}| \exp(i\Phi_{1,2})$, $n_{1,2}(T) = 2|\Psi_{1,2}|^2$ - density of superconducting electrons in different bands. Here we use notations similar to [24]. In Eqs. (1) $\phi$ means electrical scalar potential, $\Gamma_{1,2}$-relaxation time of order parameters,

*This work supported by Karabuk University BAP grant.
calculations were performed for the following values of parameters: \( T_c \). 

\( T_c \) is initially in a perfect superconducting state, is cooled through \([12]\). Results of numerical modelling presented in Fig. 1. We assume that the sample, which is the \( \kappa \) dependence two-band GL equations respectively. Before modeling we use so-called bond variables \([10]\) for the discretization of time-dependent two-band GL equations

\[
W(x, y) = \exp(iK \int^y A(\zeta, y)d\zeta),
\]

\[
V(x, y) = \exp(iK \int^y A(\zeta, \eta)d\eta).
\]

Such variables make obtained discretized equations gauge-invariant. For spatially discretization we use forward Euler method \([8,12]\). In this method we begin with partitioning the computational domain \( \Omega = [0, N_x] \times [0, N_y] \) into two subdomains, denoted by \( \Omega_{2n} \) and \( \Omega_{2n+1} \) such that

\[
\Omega_{2n} = \Omega|_{i+j=2n} \quad \text{and} \quad \Omega_{2n+1} = \Omega|_{i+j=2n+1}
\]

for \( i = 0, \ldots; N_x, j = 0, \ldots; N_y \) where \( N_x = N_x + 1, n_y = N_y + 1 \).

For numerical calculations in two-band GL theory we assume that the size of superconducting film is \( 40\lambda \times 40\lambda \), where \( \lambda \) London penetration depth of external magnetic field on superconductor \([17,11]\):

\[
\lambda^{-2}(T) = \frac{4\pi e^2}{c^2} \left( \frac{n_1(T)}{m_1} + 2\varepsilon_1(n_1(T)n_2(T))^{0.5} + \frac{n_2(T)}{m_2} \right)
\]

Under modeling we also introduce another dimensionless parameters

\[
\frac{r^\rho}{\lambda} = r^\rho; \Psi_{1,2} = \Psi_{1,2} ; A^\rho = \frac{A^\rho}{\lambda Hc\sqrt{2}}; F'(\Psi_{1,2}, A') = \frac{F(\Psi_{1,2}, A)}{\alpha_0^2 |\Psi_{1,2}|^2 + \alpha_1^2 |\Psi_{2,0}|^2}
\]

Expressions for \( \Psi_{1,2} \), and for thermodynamic magnetic field \( Hc \) are presented in \([1-7]\). The calculations were performed for the following values of parameters: \( Tc = 18K; Tc1 = 4.6K; Tc2 = 11.55K, \varepsilon_1 = -0.0976, \frac{\varepsilon_1 m_1}{\gamma_1 m_2} = 0.333 \).

For solving of corresponding discretized GL equations we will use method of adaptive grid \([12]\). Results of numerical modelling presented in Fig. 1. We assume that the sample, which is initially in a perfect superconducting state, is cooled through \( Tc \) in the absence of applied magnetic field, and then a magnetic field of an appropriate strength is suddenly turned out. Mathematically it means that, the initial state is achieved by letting \( |\Psi_{1,2}(x^\Gamma)| = 1, A_0(x^\Gamma) = 0 \) for all \( x^\Gamma \in \Omega \).

In Fig. 1, we present a contour plot of superconducting electrons. GL parameter for sample is the \( \kappa = 5 \). We can observe a partial hexagonal pattern, yet we do not observe the physically exact hexagonal pattern, as expected of homogeneous samples with uniform thickness. In some places of vortex lattice (Fig.1) coordination number changes as 5 or 7 instead exact hexagonal case 6. Similar results obtained very recently in experimental study for two-band superconductor.
LiFeAs [9]. Calculations also confirm the existence of Meissner state in thin film of two-band superconductor. It means that at fixed GL parameter $\kappa$ and external magnetic field no nucleation of vortexes of external magnetic field.

4. Conclusions

In this study we obtain time-dependent GL equations taking into account two-band character of the superconducting state. Furthermore, we perform numerical modeling of vortex lattice structure in external magnetic field in two-band superconducting films LiFeAs using GL theory. It was shown that the vortex lattice configuration in the mixed state depends upon initial state of the sample and that the system does not seem to yield hexagonal pattern for finite size homogeneous samples of uniform thickness with the natural boundary conditions. On the other hand, the time-dependent two-band GL equations leads to the expected quasi-hexagonal structure.

![Figure 1. A quasi-hexagonal vortex lattice for two-band superconductor LiFeAs.](image)

**Keywords:** Ginzburg-Landau equations, numerical modelling, vortex nucleation.

**AMS Subject Classification:** 35A08,35Q56.

**References**


NUMERICAL SOLUTIONS OF SEQUENTIAL FRACTIONAL EQUATIONS
BY USING FRACTIONAL TAYLOR BASIS

IBRAHIM AVCI

1Department of Mathematics, Eastern Mediterranean University, Turkey
e-mail: ibrahim.avci@emu.edu.tr

In this paper, a numerical method for solving the fractional equation boundary value problem of the form

\[ AD^{\alpha+1}y(t) + BD^\alpha y(t) + Cy(t) = f(t), \quad 0 \leq t \leq R, \quad 0 < \alpha \leq 1, \]
\[ y(0) = Y_0, \quad y'(0) = Y_1 \]

is given. This method is based on using fractional Taylor vector approximation. The operational matrix of the fractional integration for fractional Taylor vector is given and is utilized to reduce the solution of equation (1) to a system of algebraic equations. Illustrative examples are given to demonstrate the validity and applicability of this technique.

1. PRELIMINARIES

1.1. The Fractional Integral and Derivative. Definition 1. The Riemann-Liouville fractional integral operator of order \( \alpha \) is defined as

\[ I^\alpha y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, & \alpha > 0 \\ y(t), & \alpha = 0 \end{cases} \]  \( \alpha \in \mathbb{N} \).

Definition 2. The Caputo fractional derivative of order \( \alpha \) is defined as

\[ D^\alpha y(t) = I^{n-\alpha} \left( \frac{d^n}{dt^n} y(t) \right), \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N}. \]

1.1.1. Some Properties.

(1) The fractional integral operator and fractional Caputo’s derivative operator do not commute in general, but we have the following property:

\[ I^\alpha (D^\alpha y(t)) = y(t) - \sum_{k=0}^{n-1} y^{(k)}(0) \frac{t^k}{k!}. \]

(2) If \( s = t/R \) and \( y_1(s) = y(sR) \), then

\[ D^\alpha y(t) = \frac{1}{R^\alpha} D^\alpha y_1(s), \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N}. \]
2. Fractional Taylor approximation

2.1. Fractional Taylor Vector. The fractional Taylor vector is defined as

\[ T_m(t) = [1, t^\gamma, t^{2\gamma}, \ldots, t^{m\gamma}] \],

where \( m \) is a positive integer and \( \gamma > 0 \) is a real number.

2.2. Function Approximations. Let \( H = L^2[0,1] \), and assume that \( T_m(t) \subset H \).

\[ S = \text{span} \{ 1, t^\gamma, t^{2\gamma}, \ldots, t^{m\gamma} \} \]

and \( y \) be an arbitrary element in \( H \). Since \( S \) is a finite dimensional vector subspace of \( H \), \( y \) has a unique best approximation out of \( S \) such as \( y_0 \in S \), that is

\[ \forall \hat{y} \in S, \quad \| y - y_0 \| \leq \| y - \hat{y} \| \]

Since \( y_0 \in S \), there exists the unique coefficients \( c_0, c_1, c_2, \ldots, c_m \), such that

\[ y \simeq y_0 = \sum_{i=0}^{m} c_i t^{i\gamma} = C^T T_m(t) \] (7)

where \( T_m(t) \) is given in equation (6) and

\[ C^T = [c_0, c_1, c_2, \ldots, c_m] \] (8)

2.3. Operational matrix of integration for the fractional Taylor vector. By using equation (7), we have

\[ I^\alpha(T_m(t)) = \begin{bmatrix} 1 & \Gamma(\gamma + 1) t^{\gamma+\alpha} & \Gamma(2\gamma + 1) t^{2\gamma+\alpha} & \ldots & \Gamma(m\gamma + 1) t^{m\gamma+\alpha} \end{bmatrix} \] (9)

where \( F_\alpha \) is the operational matrix of integration and is given by

\[ F_\alpha = \text{diag} \left[ \frac{1}{\Gamma(\alpha + 1)}, \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)}, \frac{\Gamma(2\gamma + 1)}{\Gamma(2\gamma + \alpha + 1)}, \ldots, \frac{\Gamma(m\gamma + 1)}{\Gamma(m\gamma + \alpha + 1)} \right] \]

Equation (9) can be rewritten as

\[ I^\alpha(T_m(t)) = t^\alpha G_\alpha \ast T_m(t) \] (10)

where

\[ G_\alpha = \begin{bmatrix} 1 & \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + 1)} & \frac{\Gamma(2\gamma + 1)}{\Gamma(2\gamma + \alpha + 1)} & \ldots & \frac{\Gamma(m\gamma + 1)}{\Gamma(m\gamma + \alpha + 1)} \end{bmatrix} \]

and \( \ast \) denotes term by term multiplication of two matrices of the same dimensions.

Note that although equations (9) and (10) represents the same \( I^\alpha \), the representation of \( I^\alpha \) in equation (10) requires less number of operations than in equation (9).

We consider the following method to solve problem given of the form (1).

3. The numerical method

In this section, we use the fractional Taylor method for solving problem (1).

We first change the variable \( t \in [0,R] \) to \( s \in [0,1] \), by using the transformation \( s = t/R \). Now, by using property of Caputo’s derivative in (1), we get

\[ \frac{A}{R^{\alpha+1}} D^{\alpha+1} y_1(s) + \frac{B}{R^{\alpha}} D^{\alpha} y_1(s) + C y_1(s) = f_1(s) \] (11)

\[ 0 \leq s \leq 1, \quad 0 < \alpha \leq 1, \]
where \( y_1(s) = y(sR) \) and \( f_1(s) = f(sR) \). Similar to equation (7) we let \( y_1(s) \) as

\[
y_1(s) = \sum_{i=0}^{m} c_i s^{\gamma_i} = C^T T_{m\gamma}(s)
\]

(12)

where \( C^T \) is given in equation (8) and \( T_{m\gamma}(s) = [1, s^{\gamma_1}, s^{2\gamma_1}, \ldots, s^{m\gamma}]^T \).

Using the property of Caputo’s derivative and (11), we have

\[
D^\alpha \left( \frac{A}{R^{\alpha+1}} y_1'(s) + \frac{B}{R^\alpha} y_1(s) \right) + C y_1(s) = f_1(s),
\]

(13)

\[
\left( \frac{A}{R^{\alpha+1}} y_1'(s) + \frac{B}{R^\alpha} y_1(s) \right) - \left( \frac{A}{R^{\alpha+1}} y_1'(0) + \frac{B}{R^\alpha} y_1(0) \right) + C I^\alpha y_1(s) = I^\alpha f_1(s),
\]

\[
\frac{A}{R^{\alpha+1}} I^1 y_1'(s) + I^1 \left( \frac{B}{R^\alpha} y_1(s) - \frac{A}{R^{\alpha+1}} y_1'(0) - \frac{B}{R^\alpha} y_1(0) \right) + C I^\alpha f_1(s) = I^{\alpha+1} f_1(s)
\]

(14)

Substituting initial conditions and equation (12) in equation (13), we get

\[
\frac{A}{R^{\alpha+1}} (y_1(s) - y_1(0)) + \left( \frac{B}{R^\alpha} I^1 y_1(s) - \frac{A}{R^{\alpha+1}} s y_1'(0) - \frac{B}{R^\alpha} s y_1(0) \right) + C I^\alpha f_1(s) = I^{\alpha+1} f_1(s)
\]

(15)

(15)

From equations (10) and (14), we get

\[
\frac{A}{R^{\alpha+1}} (C^T T_{m\gamma}(s) - Y_0 - R s Y_1) + \frac{B}{R^\alpha} I^1 (C^T T_{m\gamma}(s) - Y_0) + C I^\alpha f_1(s) = I^{\alpha+1} f_1(s).
\]

(16)

where \( F_1(s) = I^{\alpha+1} f_1(s) \). Next, we collocate equation (15) at the equidistant nodes \( s_i = i/m \)

where \( i = 0, 1, \ldots, m \). Finally, these \( m + 1 \) algebraic equations are solved for the unknown vector \( C^T \) using well known standard methods.

**Keywords:** Caputo derivative, Riemann-Liouville fractional integral, fractional integral operator, operational matrix, fractional Taylor.

**AMS Subject Classification:** 34A08, 34A45.

**References**


ON THE SOLUTION OF THE STABILITY PROBLEM OF THREE-LAYER SYSTEMS WITH FUNCTIONALLY GRADED (FG) INTERLAYER IN THE INDUSTRIAL APPLICATIONS

A. AVEY\textsuperscript{1}, A.M. NAJAFOV\textsuperscript{2}

\textsuperscript{1}Department of Civil Engineering of Engineering Faculty of Suleyman Demirel University, Isparta, Turkey
\textsuperscript{2}Institute for Machine Elements of Azerbaijan Technical University, Baku, Azerbaijan
e-mail: li.najafov@rohe.az

Triple systems are used extensively in different areas of the industry, such as the automotive, oil, nuclear, maritime, petrochemicals and aviation sectors, where robust, rigid and lightweight construction is required. It is known that delamination problems occur at interfaces between layers of conventional triple systems \cite{1}. To overcome this problem, triple systems of functionally graded material (FGM) are proposed to gradually change the material properties in the thickness direction. FGMs belong to a new class of composite materials consisting of a mixture of ceramics and metals, characterized by a smooth and continuous change of elastic properties, and were first produced by Japanese material scientists \cite{2, 5, 6}. The superior properties of FGMs have led to their widespread use in many new industries and modern processes such as automotive, aircrafts, turbine rotors, flywheels, gears, nuclear reactors, tribology, bio-medical (implant, bone) and etc. Since mathematical modeling of such problems is very difficult, the number of publications in these matters is limited. The stability and vibration problems of the triple cylindrical shells made of Metal-FG-Ceramic (MFC) with simply-supported boundary conditions were first solved in \cite{3}. The first study on the stability behavior of pure FGM shells with mixed boundary conditions was recently proposed in \cite{4}. In this study, the stability problem of Metal-FG-Ceramic cylindrical shells (MFCCSs) with mixed boundary conditions under a lateral pressure is investigated.

Let us consider a triple system made of metal, FGM and ceramic layers under uniform lateral pressure (P). The total thickness is \( h = h_m + h_{FG} + h_c \), the radius is \( R \) and the length is \( L \) of the MFCCS, where \( h_m \) is the thickness of the outer layer \( 1 \), \( h_{FG} = 2a \) is the thickness of the FGM layer \( 2 \) and \( h_c \) is the thickness of the inner layer \( 3 \). The curvilinear coordinate system \((Oxyz)\) of the triple cylindrical shell is presented in Fig. 1.

The material properties, such as Young’s modulus and Poisson’s ratio of the MFCCS are defined as \cite{3}:

\[
(E(Z), \nu(Z)) = \begin{cases} 
E_m, \nu_m & -h/2 \leq z \leq -a, \\
E_{FG}(Z) = E_m + (E_c - E_m)\nu_c, \nu_{FG}(Z) = \nu_m + (\nu_c - \nu_m)\nu_c & -a < z < a, \\
E_c, \nu_c, & a \leq z \leq h/2,
\end{cases}
\]

where \( Z = z/h \), \( E_m, \nu_m \) and \( E_c, \nu_c \) are the Young’s modulus and Poisson’s ratio of the metal and ceramic surfaces of the FG interlayer and first and third layers, respectively.

\[104\]
The ceramic volume fraction \( V_c \) of an FG interlayer is assumed to follow a power-law distribution as [2]:
\[
V_c = (Z + 0.5)^d,
\]
where \( d \) is the volume fraction index \( (d \geq 0) \).

The stresses-strains relationships of the \( k \)th layer for MFCCSs are expressed as [3, 6]:
\[
\begin{bmatrix}
\sigma_{11}^{(k)} \\
\sigma_{22}^{(k)} \\
\sigma_{12}^{(k)}
\end{bmatrix}
= 
\begin{bmatrix}
Q_{11}^{(k)} & Q_{12}^{(k)} & 0 \\
Q_{12}^{(k)} & Q_{11}^{(k)} & 0 \\
0 & 0 & Q_{66}^{(k)}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11}^{(k)} \\
\varepsilon_{22}^{(k)} \\
\gamma_{12}^{(k)}
\end{bmatrix},
\]
where \( \sigma_{ij}^{(k)} (i, j = 1, 2) \) are the stresses, \( \varepsilon_{11}^{(k)}, \varepsilon_{22}^{(k)} \) and \( \gamma_{12}^{(k)} \) are the normal and shear strains of the MFCCSs, respectively, and \( Q_{ij}^{(k)} (i, j = 1, 2, 6) \) are the quantities depending on the normalized thickness coordinate [3].

By using basic relations, the basic equations of the MFCCSs can be derived as:
\[
\begin{bmatrix}
L_{11} & L_{12} \\
L_{21} & L_{22}
\end{bmatrix}
\begin{bmatrix}
\Phi \\
w
\end{bmatrix}
= 0,
\]
where \( L_{ij} (i, j = 1, 2, 6) \) are differential operators, depending on the MFCCSs characteristics, \( w \) and \( \Phi \) are the displacement and Airy stress function, respectively [3, 4].

At the left end of the MFCCS, i.e. for \( x = 0 \) satisfied the boundary condition \( \frac{\partial^2 w}{\partial x^2} = 0 \) and at the right end, i.e., for \( x = L \) satisfied the boundary condition \( w = \frac{\partial w}{\partial x} = 0 \), so that the solution of (4) can be described by [4]:
\[
w = w_1 \sin(0.5m_1 x) \cos(n_1 y), \quad \Phi = \Phi_1 \sin(0.5m_1 x) \cos(n_1 y),
\]
where \( w_1 \) and \( \Phi_1 \) are unknown functions, \( m_1 = \frac{m \pi}{L} \) and \( n_1 = \frac{n \pi}{R} \), in which, \( m = 1, 3, 5, \ldots \) are the wave numbers in \( x \) direction and \( n \) is the wave number in \( y \) direction.

Substituting (5) into (4) and applying Galerkin method, then eliminating \( \Phi_1 \) from the resulting system, after some mathematical operations, we obtain the following expression for the critical lateral pressure of MFCCSs under mixed boundary conditions:
\[
P_{cr}^{mix} = \frac{P_2 \times P_3 + P_1 \times P_4}{P_3 \times P_4},
\]
where \( P_j, \ (j = 1, 2, \ldots, 4, \ p) \) are coefficients depending on the MFCCSs characteristics and wave numbers [4].

The middle layer of the MFCCSs is formed of silicon nitride and stainless steel and called FGM or \( \text{Si}_3\text{N}_4/\text{SUS304} \). The effective material properties of FGM interlayer are taken from [2]. The
minimum values of $P_{mix}^{cr}$ are obtained by minimizing Eq. (6) with respect to $n (m = 1)$. The variation of $P_{mix}^{cr}$ and corresponding circumferential wave numbers for MFCCSs, MCS (metal cylindrical shell) and CCS (ceramic cylindrical shell) depending on the $R/h$ with different $h/h_{FG}$ is presented in Fig. 2. The values of $P_{mix}^{cr}$ for MFCCSs decreases slightly for the FG-linear profile, while increase for FG-quadratic and FG-cubic profiles with increasing of $h/h_{FG}$. In addition, the values of $P_{mix}^{cr}$ is decreased, as the $R/h$ increases. As the comparing the values of $P_{mix}^{cr}$ for MFCCSs with the CCS, the influences of FG interlayer on the $P_{mix}^{cr}$ decrease from 20.28% to 19.01% and decreases from 18.89% to 17.63% for FG-cubic and FG-quadratic profiles, respectively, whereas increases slightly from 16.01% to 16.13% for FG-linear profile, as $h/h_{FG}$ increases from 1.1 to 1.5 for $R/h = 100$.

![Figure 2. Variation of the $P_{mix}^{cr}$ (MPa) for MCS, CCSs and MFCCS with different profiles versus the R/h for different h/2a ($R/L = 0.5$).](image_url)

**Keywords:** Triple systems, automotive, oil, nuclear, maritime, aviation sectors.

**AMS Subject Classification:** 74E05, 74E30, 74H10, 74H55.

**References**


NUMERICAL SOLUTION TO OPTIMAL CONTROL PROBLEMS FOR LOADED DYNAMIC SYSTEMS WITH INTEGRAL CONDITIONS

K.R. AIDA-ZADE\textsuperscript{1,2}, V.M. ABDULLAYEV\textsuperscript{2,3}

\textsuperscript{1}Baku State University, \textsuperscript{2}Institute of Control Systems of NAS of Azerbaijan \textsuperscript{3}Azerbaijan State Oil and Industry University

e-mail: kamil_aydazade@rambler.ru; vaqif_ab@rambler.ru

Abstract. Optimal control problems involving non-separated multipoint and integral conditions are investigated. For numerical solution to the problem, we propose to use first order optimization methods with application of the formulas for the gradient of the functional obtained in the work. To solve the adjoint boundary problems, we propose an approach. This approach makes it possible to reduce solving initial boundary problems to solving supplementary Cauchy problems and a linear algebraic system of equations. Results of numerical experiments are given.

Keywords: Optimal control, integral conditions, multipoint conditions.

AMS Subject Classification: 49J15, 65L10.

In this paper, a numerical method of solving the optimal control problem for the system of linear (with respect to phase variables) loaded ordinary differential equations with integral and nonseparated multipoint (nonlocal) conditions is proposed. Such problems occur, for example, in the case of using the method of lines in control problems for thermal, biological, and hydrodynamic processes described by loaded partial differential equations (see \cite{3,6,9,11,12} ). Necessary optimality conditions for such problems and expressions for the gradients of the objective functionals are obtained that make it possible to use first-order optimization methods in the control space (see \cite{10} ).

For solving specific systems of differential equations for both direct and adjoint problems it is proposed to apply the operation of successive shift of the problem conditions to the load points that was proposed in \cite{4}; this operations elaborates the idea of transferring the boundary conditions (see \cite{5} ). Schemes for the numerical solution of the problems under consideration are proposed, and the results of numerical experiments are presented.

Let us consider the following optimal control problem for the process described by a system, which is linear with respect to the phase variable, of loaded ordinary differential equations:

\begin{equation}
\dot{x}(t) = A(t,u)x(t) + \sum_{s=1}^{l_3} B^s(t)x(\tilde{t}_s) + C(t,u) , \quad t \in [t_0, T],
\end{equation}

where $x(t) \in E^n$ is the phase variable; $u(t) \in U \subset E^r$ is the control vector-function from the class of piecewise continuous functions, admissible values of which belong to the given compact set $U$; the $(n \times n)$ matrix functions $A(t,u) \neq \text{const}$, $B^s(t)$, $s = 1, 2, ..., l_3$, and $n-$dimensional vector-function $C(t,u)$ are continuous with respect to $t$ and continuously differentiable with respect to $u$. The points of loading time $t_s \in [t_0, T]$; $\tilde{t}_{s+1} > t_s$, $s = 1, 2, ..., l_3$ are given.
Nonseparated multipoint and integral conditions are given in the following form:

\[
\sum_{i=1}^{l_1} \int_{t_{2i-1}}^{t_{2i}} \tilde{D}_i(\tau)x(\tau)d\tau + \sum_{j=1}^{l_2} \tilde{D}_jx(\tilde{t}_j) + \sum_{s=1}^{l_3} \tilde{D}_s x(\tilde{t}_s) = L_0, \tag{2}
\]

where the continuous matrix function \(\tilde{D}_i(\tau)\) and scalar matrices \(\tilde{D}_j, \tilde{D}_s\) have the dimension \((n \times n)\); \(L_0\) is the n-dimensional vector; \(\tilde{t}_i, \tilde{t}_j\) are the points of time belonging to \([t_0, T]\); \(\tilde{t}_{i+1} > \tilde{t}_i, \tilde{t}_{j+1} > \tilde{t}_j\), \(i = 1, 2, ..., 2l_1 - 1, j = 1, 2, ..., l_2 - 1, l_1, l_2, l_3\) are given.

To be definite, without loss of generality, make an assumption that

\[
\min \left(\tilde{t}_1, \tilde{t}_1\right) = t_0, \quad \max \left(\tilde{t}_{l_1}, \tilde{t}_{l_2}, \tilde{t}_{l_2l}\right) = T, \tag{3}
\]

and for all \(i = 1, 2, ..., 2l_1, j = 1, 2, ..., l_2, s = 1, 2, ..., l_3\), the following condition holds

\[
\tilde{t}_j, \tilde{t}_s \in [\tilde{t}_{2i-1}, \tilde{t}_{2i}]. \tag{4}
\]

The target functional is as follows:

\[
J(u) = \Phi(x(\tilde{t})) + \int_{t_0}^{T} f^0(x, u, t)dt \to \min_{u(t) \in U}, \tag{5}
\]

where the function \(\Phi\) is continuous with respect to its arguments along with the private derivatives, and \(f^0(x, u, t)\) is continuously differentiable with respect to \((x, u)\), and continuous with respect to \(t\); \(\tilde{t} = (\tilde{t}_1, \tilde{t}_2, ..., \tilde{t}_{2l_1+l_2})\) is the ordered union of points of the sets \(\tilde{t} = (\tilde{t}_1, \tilde{t}_2, ..., \tilde{t}_{l_1}), \tilde{t} = (\tilde{t}_1, \tilde{t}_2, ..., \tilde{t}_{l_2}, ..., \tilde{t}_{l_2l})\), \(i = 1, 2, ..., l_1, j = 1, 2, ..., l_2, l_1 + l_2 + l_3 - 1\).

Suppose that the problem (1) and (2) is solvable under any admissible control \(u(t) \in U \in \mathbb{R}^r\).

**Theorem.** The gradient of the functional in the problem (1)-(5) is determined as follows:

\[
(\text{grad} J(u))^* = \frac{\partial f^0(x, u, t)}{\partial u(t)} - \psi^*(t) \left[ \frac{\partial A^*(t, u)}{\partial u(t)} x(t) + \frac{\partial C^*(t, u)}{\partial u(t)} \right]. \tag{6}
\]

where the vector-function \(\psi(t) \in \mathbb{R}^n\) and the vector \(\lambda \in \mathbb{R}^n\) satisfy the following differential equation:

\[
\dot{\psi}(t) = -A^*(t, u) \psi(t) - \sum_{s=1}^{l_3} \delta \left( t - \tilde{t}_s \right) \int_{t_0}^{T} B^*(t) \psi(t) dt + \sum_{i=1}^{l_1} \left[ \chi(\tilde{t}_i) - \chi(\tilde{t}_{2i-1}) \right] \tilde{D}_i^*(t) \lambda + \frac{\partial f^0(x, u, t)}{\partial x(t)} \tag{7}
\]

the following boundary conditions

\[
\psi(t_0) = \left\{ \begin{array}{l}
\left( \frac{\partial \Phi(x(t))}{\partial x_1} \right)^* + \tilde{D}_1 \lambda, \quad \text{for } t_0 = \tilde{t}_1, \\
\left( \frac{\partial \Phi(x(t))}{\partial x_1} \right)^* + \tilde{D}_1 \lambda, \quad \text{for } t_0 = \tilde{t}_1, \\
\left( \frac{\partial \Phi(x(t))}{\partial x_1} \right)^* , \quad \text{for } t_0 = \tilde{t}_1, \\
\end{array} \right. \tag{8}
\]

\[
\psi(T) = \left\{ \begin{array}{l}
- \left( \frac{\partial \Phi(x(t))}{\partial x_1} \right)^* - \tilde{D}_t \lambda, \quad \text{for } \tilde{t}_{l_3} = T, \\
- \left( \frac{\partial \Phi(x(t))}{\partial x_1} \right)^* - \tilde{D}_t \lambda, \quad \text{for } \tilde{t}_{l_3} = T, \\
- \left( \frac{\partial \Phi(x(t))}{\partial x_1} \right)^* \lambda, \quad \text{for } \tilde{t}_{l_3} = T, \\
\end{array} \right. \tag{9}
\]
the following jump conditions at the intermediate points $\tilde{t}_j$, for which $t_0 < \tilde{t}_j < T$,
\[
\psi^+ (\tilde{t}_j) - \psi^- (\tilde{t}_j) = \left( \frac{\partial \Phi(x(\hat{t}_j))}{\partial x(\hat{t}_j)} \right)^* + \tilde{D}_j^\lambda, \quad j = 1, 2, \ldots, l_2, \tag{10}
\]
the following jump conditions at the loading points $\tilde{t}_s$, for which $t_0 < \tilde{t}_s < T$,
\[
\psi^+ (\tilde{t}_s) - \psi^- (\tilde{t}_s) = \left( \frac{\partial \Phi(x(\hat{t}_s))}{\partial x(\hat{t}_s)} \right)^* + \tilde{D}_s^\lambda, \quad s = 1, 2, \ldots, l_3, \tag{11}
\]
and the following jump conditions at the points $\tilde{t}_i, i = 1, 2, \ldots, 2l_1$, for which $t_0 < \tilde{t}_i < T$,
\[
\psi^+ (\tilde{t}_i) - \psi^- (\tilde{t}_i) = \left( \frac{\partial \Phi(x(\hat{t}_i))}{\partial x(\hat{t}_i)} \right)^*, \quad i = 1, 2, \ldots, 2l_1. \tag{12}
\]
Here "*" is the transposition sign; $\delta(\cdot)$ is the delta function; $\chi(t)$ is the Heaviside function.

For numerical solution to the problem, we propose to use standard procedures of first order optimization. To determine the value of the gradient by the formula (6), at each iteration, it is necessary to: 1) solve the problem (1) under the current control with multipoint and integral conditions (3) using the technique of convolving integral conditions into local conditions (here we mean to use the results of the work [7]); 2) solve the adjoint problem (7)-(12) using the generalized operation of iterated shifts, making a special emphasis on the participation of the parameters $\lambda$ in the conditions (10)-(1) (here we mean to use the results of the works [2, 4]). Following the shift of the conditions, we obtain an algebraic system of equations with the $n(l_3 + 2)$ unknowns $\lambda$, with the values of the phase trajectory at one of two ends of the interval, and with the loading points [1, 4, 8].

Results of numerical experiments obtained by solving the problems of the form (1)-(5) are given in the presentation.

REFERENCES

NUMERICAL SOLUTION TO OPTIMAL CONTROL FOR WAVE PROCESS WITH THE SET OF INITIAL CONDITIONS

K.R. AIDA-ZADE\textsuperscript{1}, Y.R. ASHRAFOVA\textsuperscript{1}

\textsuperscript{1}Baku State University, Institute of Control Systems of ANAS, Baku, Azerbaijan

e-mail: kamil_aydazade@rambler.ru ashrafova_yegana@yahoo.com

1. Problem statement

In this paper, we propose an approach to the optimal control of processes, described by differential equations of hyperbolic type with a set of initial conditions. The influence of initial conditions on the current state of the process weakens over time, taking into account the participation of dissipative terms in real processes, which characterize the resistance of internal or external nature. There are also possible cases, when it is impossible to measure the initial state of the process accurately for some reason. In this case, we have an information about the set of possible initial states depending on the parameters from the given parametric set and it is necessary to control the state of the process without knowing the exact information about the initial state of the process.

First A.N. Tikhonov studied the boundary value problems without initial conditions for differential equations of parabolic and hyperbolic types \cite{11}, and offered the method of study the boundary-value problems without initial conditions, gave their first strict solutions. In he proved the uniqueness of the solution to the problem without initial conditions for heat equation.

In the paper, we study optimal control of processes, described by differential equations of hyperbolic type without accurate information about the initial state of the process. Such problems arise at controlling of real long time functioning evolutionary processes, the value of initial state of the processes which doesn’t impact to their current state. A practical example of such problem is the optimal control of pipeline transportation of hydrocarbon raw material \cite{1, 2, 5, 9, 12}. There are also possible cases, when it is impossible to measure the initial state of the process accurately for some reason. In this case, we have an information about the set of possible initial states depending on the parameters from the given parametric set and it is necessary to control the state of the process without knowing the exact information about the initial values of the state of process \cite{2–10}. In this paper, we propose an approach to the optimal control of processes, described by differential equations of hyperbolic type with a set of initial conditions.

Let’s consider the wave process, described by the following system of differential equations of hyperbolic type \cite{9, 12}:

\[
\begin{align*}
-\frac{\partial P(x,t)}{\partial x} &= \frac{\rho}{S} \frac{\partial Q(x,t)}{\partial t} + a \xi Q(x,t), \quad t \in (0,T), \quad x \in (0,l), \\
-\frac{\partial P(x,t)}{\partial t} &= c^2 \rho \frac{\partial Q(x,t)}{\partial x}, \\
P(0,t) &= v_0(t), \quad P(l,t) = v_l(t), \quad t \in (0,T),
\end{align*}
\]  

(1)

where $P(x,t)$, $Q(x,t)$ are the phase state of the process (for instance, the pressure and the rate of transported raw materials in the point $x$ of the pipeline at the moment of time $t$), determined from the solutions to the system 1-2 for corresponding admissible value of optimizing control vector-function $v = (v_0(t), v_l(t))$, $t \in [0,T]$; $a$ – is the friction coefficient. If the wave process 1,
which depend on the parameters from \( D \), where \( D \) set \( \rho \) and the density function conditions. The control functions belong to the set with density functions of distribution \( \rho \), i.e. the values \( t \) and \( \upsilon \) boundary controls \( T > 0 \) the regime where it differs as little as possible from a given state, to the moment of time of transported raw materials respectively, by which the pumping station can operate.

Information on the investigating process it is known that the functions determining the possible initial state of the process are not given accurately, but on the basis of a priori determination initial state of the process are no given accurately, but on the basis of a priori information the functions determining the possible initial state of the process, belong to some admissible set

\[
Q_0(x) = Q_0(x, \gamma^0) \in L_2[0, l], \quad P_0(x) = P_0(x, \gamma^p) \in L_2[0, l], \quad x \in [0, l],
\]

which depend on the parameters \( \gamma^p, \gamma^q \in D \subset \mathbb{R}^r \):

\[
D = \{ \gamma = (\gamma^p, \gamma^q) \in \mathbb{R}^r : \frac{\gamma^p}{\gamma^q} \leq \gamma^p \leq \frac{\gamma^q}{\gamma^q} \leq \gamma^q \leq \gamma^q \leq \gamma^q, \quad i = 1, ..., r \}
\]

and the density function \( \rho_D(\gamma) \) is given. It is possible the existence of a set \( D_N \), instead of the set \( D \), which is determined with a certain finite number of functions

\[
\varphi(x; \gamma_i) = (Q_0(x; \gamma_i^0), P_0(x; \gamma_i^p)), \quad x \in [0, l], \quad i = 1, 2, ..., N.
\]

which depend on the parameters from \( D_N = \{ \gamma_i : \gamma_i^p \leq \gamma_i^p \leq \gamma_i^p \leq \gamma_i^q \leq \gamma_i^q \leq \gamma_i^q \leq \gamma_i^q \leq \gamma_i^q \leq \gamma_i^q, \quad i = 1, ..., N \} \) with density functions of distribution \( \rho_D(\gamma_i) \), \( i = 1, ..., N \). We assume the restrictions on the optimized values of the boundary conditions on the basis of technological and technical conditions. The control functions belong to the set

\[
V = \{ v(t) = (v_0(t), v_1(t)) \in L_2[0, T] : v_0 \leq v_0(t) \leq \bar{v}_0, v_1 \leq v_1(t) \leq \bar{v}_1 \text{ a.e. on } [0, T] \},
\]

where \( v_0, v_1, \bar{v}_0, \bar{v}_1 \) are the given upper and lower admissible values of the pressure and the rate of transported raw materials respectively, by which the pumping station can operate.

It is required, the wave process characterized by pressure and rate of raw material to lead the regime where it differs as little as possible from a given state, to the moment of time \( T > 0 \), by controlling boundary conditions. The problem consists of finding such values of boundary controls \( v_0(t), v_1(t), t \in (0, T] \), in which the functional gets it’s minimum value defined particularly in the following form:

\[
J(v) = \int_D I(v, \varphi) \rho_D(\gamma) d\gamma \rightarrow \min_{v \in V}
\]

\[
I(v, \varphi) = \int_0^T \left( (Q(x, T; v, \varphi) - q_T(x))^2 + (P(x, T; v, \varphi) - p_T(x))^2 \right) dx + \alpha \| v(t) \|^2_{L_2^2[0, T]}.
\]

Here \( \| \cdot \|_{L_2^2[0, T]} \) is the norm of vector-function; \( \alpha > 0 \) is a weight coefficient. The functional (7) determines the assessment of the mean value of deviation of the state of the process at \( t = T \) from the desired state of \( (q_T(x), p_T(x)) \) for all possible initial conditions \( (Q_0(x, \gamma), P_0(x, \gamma)), \gamma \in D \); \( \rho_D(\gamma) \) is the given density function of the initial values distributions on the set of \( D \).

In the study of the boundary-value and optimal control problems the interval \([t_0, T]\) plays an important role, when the state of the process is almost does not depend on the values of the initial conditions at \( t = 0 \).
2. Description of approach to the solution of the problem

We suggest using the iterative methods of optimization of the first order for the numerical solution of optimal control problems, based on the use of analytical formulas derived below for control functions of the target functional gradient. Next we obtain the formulas for the gradient $\text{grad} \ J(v; \varphi)$ of the functional for any one arbitrary selected admissible initial condition $\varphi$ of the considered problem. The formulas for the components of the gradient of target functional for control functions $v_0(t), v_l(t)$ are determined in the following form for the considered problem:

$$\text{grad}_{v_0(t)} J(v, \varphi) = - \int_D (\psi_1(0, t) + 2\alpha \nu_0(t)) \rho_D(\gamma) d\gamma, \ t \in [0, T],$$

$$\text{grad}_{v_l(t)} J(v, \varphi) = - \int_D (\psi_1(l, t) + 2\alpha \nu_l(t)) \rho_D(\gamma) d\gamma, \ t \in [0, T].$$

To carry out computer experiments on numerical investigation of optimal boundary control problems with inaccurately given initial conditions, we consider the following test problem for the example of controlling the process of fluid flow in a linear section of the main pipeline, described by a system of differential equations of hyperbolic type (1). The obtained results can find application in the studies related with the controlling of many long-time functioning processes with distributed parameters, described by the systems of partial differential equations.

**Keywords:** Optimal control, set of initial conditions, differential equations of hyperbolic type, fluid flow, pipeline.

**AMS Subject Classification:** 35Q35, 49K20, 65M32. 65M55.

**References**


SPECTRAL PROPERTIES AND SCATTERING PROBLEMS OF EIGENPARAMETER DEPENDENT DISCRETE IMPULSIVE STURM-LIOUVILLE EQUATIONS

YELDA AYGAR¹, ELGIZ BAIRAMOV¹, GULÇEHRE OZBEY¹

¹Ankara University, Department of Mathematics, 06100 Tandogan, Ankara, Turkey
e-mail: yaygar@ankara.edu.tr

Scattering theory is an important research area in mathematical physics. Analysis of scattering problems have meanings in a lot of branches of physics. Because scattering experiments play a crucial role in determining the structure of matter. In this way, scattering problems have an interest role due to their applications in mathematics, mathematical physics and natural sciences. There are many books devoted exclusively to scattering theory. Because of this importance spectral analysis and scattering problems of differential and difference equations became a popular topic for mathematicians. Many authors have investigated the scattering analysis of boundary value problems generated with differential and discrete equations ([1, 7–9, 17, 18, 20]).

In [20], the author has investigated the scattering problem of the Sturm-Liouville boundary value problem

\[-y'' + q(x)y = \lambda^2 y, \quad 0 \leq x < \infty,\]
\[y(0) = 0,\]

where \(q\) is a real-valued function and \(\lambda\) is a spectral parameter and

\[\int_0^\infty x|q(x)|dx < \infty.\]

It is shown in [6] that under the condition (3), the equation (1) has a solution \(e(x, \lambda)\) satisfying the condition

\[\lim_{x \to \infty} e(x, \lambda)e^{-i\lambda x} = 1, \quad \lambda \in \mathbb{C}_+ := \{\lambda \in \mathbb{C} : \lambda \geq 0\}.\]

\(e(x, \lambda)\) is called the Jost solution of (1). The Jost solution is analytic with respect to \(\lambda\) in \(\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \lambda > 0\}\) and continuous up to the real axis. In [6], the author also gives the integral representation of \(e(x, \lambda)\) as

\[e(x, \lambda) = e^{i\lambda x} + \int_x^\infty K(x, t)e^{i\lambda t}dt, \quad \lambda \in \mathbb{C}_+\]

where the kernel \(K(x, t)\) may be expressed in terms of the potential function \(q\). The function

\[e(\lambda) := e(0, \lambda) = 1 + \int_0^\infty K(0, t)e^{i\lambda t}dt, \quad \lambda \in \mathbb{C}_+ := \{\lambda \in \mathbb{C} : \lambda \geq 0\}\]

is called the Jost function of (1)-(2) and \(e(\lambda)\) has a finite number of simple zeros in \(\mathbb{C}_+\). They lie on the imaginary axis. Let \(i\lambda_k, k = 1, 2, \ldots, n\) be the zeros of the Jost function \(e(\lambda)\), numbered
in the order of increase of their module \((0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n)\), and let \(m_k^{-1}\) be the norm of the function \(e(x, i\lambda_k)\) in \(L_2(0, \infty)\), i.e.,

\[
m_k^{-2} = \int_0^\infty e^2(x, i\lambda_k) dx, \quad k = 1, 2, \ldots, n,
\]

and

\[
S(\lambda) := \frac{e(\lambda)}{\overline{e(\lambda)}}, \quad \lambda \in (-\infty, \infty).
\]

The function \(S(\lambda)\) is the scattering function of (1)-(2). The collection

\[
\{S(\lambda), \lambda \in (-\infty, \infty); \lambda_k, m_k, k = 1, 2, \ldots, n\}
\]

(6)

is the scattering data of boundary value problem (1)-(2). When the potential function \(q\) is given, the problem of finding scattering data (6) and learning the properties of scattering data is the direct problem for scattering theory. Conversely, the problem of finding the potential function \(q\) according to the scattering data given in (6) is an inverse problem of scattering theory. Several authors have studied about direct and inverse scattering problems for Sturm–Liouville equations, discrete Sturm–Liouville equations, Schrödinger and Dirac equations (\[4\]-\[6\], \[10\]-\[16\], \[19\]).

In \[11\] and \[14\] the authors have given the necessary and sufficient conditions for the existence of the solution of inverse scattering problem for a discrete one-dimensional Schrödinger equation on whole axis and on the semi axis. Similar problems for continuous Schrödinger equation has been thoroughly studied in \[5\], \[16\], \[4\]. In \[15\], the authors have investigated the properties of eigenvalues of the discrete Sturm-Liouville boundary value problem. Furthermore, the authors give the scattering solutions and Jost solution of (1)-(2) with impulsive conditions in \[2\], \[3\]. They investigate the properties of scattering function and the properties of eigenvalues of this problem.

All mentioned studies given in literature about scattering problems do not consist impulsive conditions except \[2\] and \[3\]. In this work, we will interested in a boundary value problem for discrete Sturm-Liouville equation with impulsive condition. This study differs from \[3\] that it contains spectral parameter in boundary condition and because of this it becomes more applicable in many parts of physics, mathematics and other disciplines including engineering, economics, biology.

Let us consider an impulsive discrete Sturm-Liouville boundary value problem generated by following difference equation

\[
a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N} \setminus \{m_0 - 1, m_0, m_0 + 1\}
\]

(7)

with the boundary condition

\[
(\mu_0 + \lambda \mu_1) y_1 + (\nu_0 + \lambda \nu_1) y_0 = 0,
\]

(8)

and the impulsive conditions

\[
y_{m_0+1} = \gamma_1 y_{m_0-1},
\]

\[
y_{m_0+2} = \gamma_2 y_{m_0-2}, \quad \gamma_2 \neq 0, \quad \gamma_1, \gamma_2 \in \mathbb{R}
\]

(9)

where \(\lambda = 2 \cos z\) is a spectral parameter, \(\mu_0, \mu_1, \nu_0, \nu_1 \in \mathbb{R}\), \(\{a_n\}_{n \in \mathbb{N} \cup \{0\}}\) and \(\{b_n\}_{n \in \mathbb{N}}\) are real sequences satisfying the condition

\[
\sum_{n \in \mathbb{N}} n (|1 - a_n| + |b_n|) < \infty.
\]

(10)

Through the work, we will assume that \(a_n \neq 0\) for all \(n \in \mathbb{N} \cup \{0\}\). First, we find the Jost solution and scattering solutions of (7)-(9) and by using the properties of these solutions, we get the scattering function, Green function and resolvent operator of boundary value problem (7)-(9). Secondly, we investigate the properties of scattering function of the boundary value problem and give an asymptotic equation to get the properties of eigenvalues and continuous
spectrum of (7)-(9). Finally, we present an example of a different discrete impulsive boundary value problem which is a special case of (7)-(9). Discussing the properties of Jost solution and scattering function of this example, we determine the region of eigenvalues and continuous spectrum of this problem.

Keywords: Scattering solution, resolvent, green function, impulsive condition, eigenvalue, continuous spectrum.

AMS Subject Classification: 34L05, 34L25, 34K10.

REFERENCES

HAAR WAVELET METHOD FOR NUMERICAL SOLUTION OF PANTOGRAPH FUNCTIONAL DIFFERENTIAL EQUATIONS

IMRAN AZIZ, SHUMAILA YASMEEN, ROHUL AMIN

Department of Mathematics, University of Peshawar, Pakistan
e-mail: imran_aziz@uop.edu.pk

Abstract. In this paper a new method is presented whose purpose is to solve pantograph functional differential equations of second-order with boundary conditions with the help of Haar wavelet collocation method. The proposed method is simple, accurate and the algorithms obtained using the proposed method are more efficient. The proposed method is tested upon many benchmark problems present in the existing literature and the computational results confirm that the proposed method is efficient and accurate. The computational results are equated with analytic solutions in order to test the performance of the proposed method. We have calculated the Maximum Absolute Errors (MAE) as well as experimental rates of convergence. The computational results show that the proposed method is simple, robust, accurate and efficient.

Keywords: Pantograph equations, boundary value problems, haar wavelet, collocation method.

AMS Subject Classification: 34K28, 65L03.

1. Introduction

The class of pantograph FDEs is a subfamily of the class of delay differential equations. The name pantograph is due to the work of Ockendon and Tayler [4]. Researchers have started working on numerical solution of various families of Pantograph FDEs (PFDEs) in the last decade. In this regard, Taylor polynomials and the stability of special classes of Runge-Kutta methods [3] have been thoroughly investigated. Trapezoidal rule and its asymptotic properties has also been discussed in [2]. The authors in [5] have used Bessel functions of the first kind with unknown coefficients for solving the system of multi-pantograph equations with mixed conditions. A general reference for the numerical analysis of the pantograph equation and other delay equations is the book [1].

In the present work, we will apply HWCM for numerical solution of second order pantograph differential equations. Haar Wavelet (HW) is the simplest wavelet having compact support. Haar functions are piecewise constant functions defined on a compact interval. Due to constant nature of Haar functions, their derivatives vanish over the entire interval and hence in the HW collocation method the direct approach, in which the unknown function is approximated, can not be applied. In the indirect approach, the highest derivative of the unknown function in the differential equation is approximated using HW approximation and the approximations for the lower derivatives as well as the unknown function are obtained using integration.
2. Computational method

For second order linear or nonlinear initial or BVP, initially we suppose that \( \omega''(s) \) is a function belonging to the class of square integrable functions. Due to this reason this function can be approximated using Haar wavelet expansion which is given as under.

\[
\omega''(s) = \sum_{i=1}^{N} \alpha_i \text{haar}_i(s). \tag{1}
\]

The above equation is integrated with limits 0 to \( s \) to obtain Haar approximations for \( \omega'(s) \) and \( \omega(s) \) with the help of BCs. These expressions are then substituted in given equation and after this substitution collocations are inserted in the given FDE which results in a system of linear or nonlinear equations. This system is linear if the given equation is linear and nonlinear if the given equation is nonlinear. This system is solved with the help of Gauss elimination method in case of linear problems or using iterative method (Newton’s method or Broyden’s method) in case of nonlinear problem. After having solution of this system, Haar coefficients \( \alpha_i \)'s, \( i = 1, 2, \ldots, N \) are obtained. The approximate solution if finally obtained using these Haar coefficients \( \alpha_i \)'s, \( i = 1, 2, \ldots, N \). The details of the method for different BCs as well as linear and nonlinear problems is given below.

Consider second order FDE defined on \([0, 1]\) interval

\[
\omega''(s) = \phi_1(s) \omega'(\mu_1 s) + \phi_2(s) \omega(\mu_2 s) + \phi_3(s), \quad 0 < \mu_1, \mu_2 \leq 1. \tag{2}
\]

In above equation \( \phi_1(s), \phi_2(s) \) and \( \phi_3(s) \) are given functions.

Let us suppose that

\[
\omega''(s) = \sum_{i=1}^{N} \alpha_i \text{haar}_i(s), \tag{3}
\]

Integration of (3) produces the following approximation with the help of BCs

\[
\omega'(s) = (\eta_2 - \eta_1) + \sum_{i=1}^{N} \alpha_i (\text{ihar}_1(s) - \text{ihar}_2(b)), \tag{4}
\]

\[
\omega(s) = \eta_1 + t(\eta_2 - \eta_1) + \sum_{i=1}^{N} \alpha_i (\text{ihar}_2(s) - t\text{ihar}_2(b)). \tag{5}
\]

In linear case, the above expressions are substituted in given linear equation to obtain

\[
\sum_{i=1}^{2M} \alpha_i (\text{haar}_1(s) - \phi_1(s) (\text{ihar}_1(\mu_1 s) - \text{ihar}_2(b)) - \phi_2(s) (\text{ihar}_2(\mu_2 s) - \mu_2 s)) = \phi_1(s) (\eta_2 - \eta_1) + \phi_2(s) (\eta_1 + \mu_2 t (\eta_2 - \eta_1)) + \phi_3(s). \tag{6}
\]

We substitute the collocation point in the above equation to obtain

\[
\sum_{i=1}^{2M} \alpha_i (\text{haar}_1(s_j) - \phi_1(s_j) (\text{ihar}_1(\mu_1 s_j) - \text{ihar}_2(b)) - \phi_2(s_j) (\text{ihar}_2(\mu_2 s_j) - \mu_2 s_j)) = \phi_1(s_j) (\eta_2 - \eta_1) + \phi_2(s_j) (\eta_1 + \mu_2 t (\eta_2 - \eta_1)) + \phi_3(s_j). \tag{7}
\]

Using matrix notations, we have

\[
XA = Y.
\]
3. Numerical experiments

We will use $L_\infty$ as a notation for the MAE at $N$ collocation points. The experimental rate of convergence $R_c$ are also computed for test problems which is mathematically defined as

$$R_c(N) = \frac{\log[E_c(N/2)/E_c(N)]}{\log 2}. \quad (8)$$

**Example 1.** Consider the following first order pantograph delay differential equation:

$$\omega'(s) = \frac{1}{2} e^{0.5s} u\left(\frac{s}{2}\right) + \frac{1}{2} \omega(s), \quad 0 \leq t \leq 1, \quad (9)$$

$$u(a) = 1,$$

The analytic solution is given by

$$\omega(s) = e^s. \quad (10)$$

Fig. 3 shows the comparison of analytic solution with Haar solution using $N = 32$ number of collocation points. One can observe from the figure that the performance of the method can be strengthened by adding more collocation points to the existing set of collocation points.

4. Conclusion

First and second order pantograph differential equations are considered. Numerical solution of these equations are approximated using HWCM. Different types of BCs are considered. An important advantage of HWCM is that the method can be applied to initial- as well as BVPs having different types of BCs with slight modification.

**References**

NUMERICAL SOLUTION OF A VOLTERRA INTEGRAL-ALGEBRAIC EQUATIONS

AZIZOLLAH BABAKHANI

Department of Mathematics, Babol Noshirvani University of Technology, Iran
e-mail: babakhani@nit.ac.ir

Integral algebraic equations (IAEs) are not as well known as integral equations or differential algebraic equations (DAEs). But there are some major reasons that make their investigations important. One of these is that a DAE problem can be considered an IAE problem. The second one is that IAEs are more general than integral equations of the first and second kinds and so on.

Here we consider an IAE of the form

\[ A(s)X(s) - \int_0^s K(s,t)X(t)dt = Y(s), \quad 0 \leq s \leq 1 \]  

(1)

where

\[ A(s) = [a_{pq}(s)], \quad p, q = 1, 2, \cdots, m, \]
\[ X(s) = [x_1(s) \; x_2(s) \; \cdots \; x_m(s)]^T, \]
\[ Y(s) = [y_1(s) \; y_2(s) \; \cdots \; y_m(s)]^T, \]
\[ K(s,t) = [k_{pq}(s,t)], \quad p, q = 1, 2, \cdots, m. \]

In this system \( A, Y \) and \( K \) are given functions and \( X \) is the solution to be determined. If \( \det A(s) = 0 \), this system is called as Volterra Integral-Algebraic Equations (IAEs). Under the condition \( \det A(s) = 0 \), the system can have several solutions or no solution at all. Sufficient conditions for the existence of unique continuous solution have been presented in [1].

**Theorem 1.** [1] Assume that the system (1) with \( \det A(s) = 0 \) satisfies the following conditions:

1. \( \text{Rank} A(s) = \text{deg} (\det [\lambda A(s) + K(s,s)]) = k \), for all \( s \in [0, 1] \), where \( k \) is constant and \( \lambda \) is a scaler.
2. \( \text{Rank} A(0) = \text{Rank} [A(0)|X(0)]. \)
3. \( A(s) \in C^1_{[0,1]}, X(s) \in C^1_{[0,1]} \) and \( K(s,t) \in C^1_{\triangle} \), where \( \triangle = \{0 \leq t \leq s \leq 1\} \).

Then the system has a unique continuous solution. System (1) has been widely applied in engineering and physics, particularly, it arises in a number of important problems of the theory of elasticity, neutron transport, and scattering of particles, see for examples [2, 3, 4].

In this paper, we study a numerical solution for Volterra integral-algebraic equations. Taylor expansion method for solving such equations numerically is developed and an error analysis for the proposed method is provided.

We consider the system of linear Volterra equations with variable coefficients (1). This system also can be written as

\[ \sum_{q=1}^m a_{pq}(s)x_q(s) - \sum_{q=1}^m \int_0^s k_{pq}(s,t)x_q(t)dt = y_p(s), \quad 0 \leq s \leq 1, \]  

(2)
where \( p = 1, 2, \ldots, m \).

Let \( k_{pq}^{(i)}(s, t) = \frac{\partial^i k_{pq}(s, t)}{\partial s^i}, \) \( i = 1, 2, \ldots, n \) and \( k_{pq}^{(i)}(s, s) = \frac{\partial^i k_{pq}(s, t)}{\partial s^i}|_{t=s} \). Also, let \( [k_{pq}^{(i)}(s, s)]^{(j)} = \frac{\partial^j k_{pq}^{(i)}(s, s)}{\partial s^j} \), \( i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, m \). Differentiating (2) \( n \) times, we obtain

\[
\sum_{q=1}^{m} \left[ a_{pq}'(s) x_q(s) + a_{pq}(s) x_q'(s) - k_{pq}(s, s) x_q(s) - \int_{0}^{s} k_{pq}'(s, t) x_q(t) \, dt \right] = y_p'(s) \tag {3}
\]

\[
\sum_{q=1}^{m} \left[ a_{pq}''(s) x_q(s) + 2a_{pq}'(s) x_q'(s) + a_{pq}(s) x_q''(s) - k_{pq}'(s, s) x_q(s) - k_{pq}(s, s) x_q'(s) - k_{pq}'(s, s) x_q(s) - \int_{0}^{s} k_{pq}''(s, t) x_q(t) \, dt \right] = y_p''(s) \tag {4}
\]

\[
\vdots \quad \vdots \quad \vdots
\]

And by applying successively \( n \) times the Leibnitz’s rule (dealing with differentiation of products and differentiation of integrals), we have

\[
\sum_{q=1}^{m} \sum_{j=0}^{n} \binom{n-1}{j} k_{pq}^{(n-1-j)}(s, s) x_q^{(j)}(s) - \sum_{j=0}^{n-1} \binom{n-1}{j} k_{pq}^{(n-1-j)}(s, s) x_q^{(j)}(s)
\]

\[
- \sum_{i=0}^{n-2} \sum_{j=0}^{n-2-i} \binom{n-2-i}{j} [k_{pq}^{(i+1)}(s, s)]^{(n-2-i-j)} x_q^{(j)}(s)
\]

\[
- \int_{0}^{s} k_{pq}^{(n)}(s, t) x_q(t) \, dt \right] = y_p^{(n)}(s). \tag {5}
\]

For each arbitrary but fixed \( s \),

\[
x_q(t) \approx x_q(s) + x_q'(s)(t-s) + \cdots + \frac{1}{n!} x_q^{(n)}(s)(t-s)^n. \tag {6}
\]

Substituting (6) into (2) and into each of equations (3), (4) and (5), we obtain

\[
\sum_{q=1}^{m} \left[ a_{pq}(s) x_q(s) - \sum_{j=0}^{n} \frac{1}{j!} \int_{0}^{s} k_{pq}(s, t)(t-s)^j \, dt \right] x_q^{(j)}(s) = y_p(s), \tag {7}
\]

\[
\sum_{q=1}^{m} \left[ a_{pq}'(s) x_q(s) + a_{pq}(s) x_q'(s) - k_{pq}(s, s) x_q(s) - \sum_{j=0}^{n} \frac{1}{j!} \int_{0}^{s} k_{pq}'(s, t)(t-s)^j \, dt \right] x_q^{(j)}(s) = y_p'(s) \tag {8}
\]

\[
\vdots \quad \vdots \quad \vdots
\]

\[
\sum_{q=1}^{m} \left[ a_{pq}''(s) x_q(s) + 2a_{pq}'(s) x_q'(s) + a_{pq}(s) x_q''(s) - k_{pq}'(s, s) x_q(s) - k_{pq}(s, s) x_q'(s) - k_{pq}'(s, s) x_q(s) - \int_{0}^{s} k_{pq}''(s, t) x_q(t) \, dt \right] x_q^{(j)}(s) = y_p''(s) \tag {9}
\]

\[
\vdots \quad \vdots \quad \vdots
\]
and
\[ \sum_{q=1}^{m} \left[ \sum_{j=0}^{n} \binom{n}{j} a_{pq}^{(n-j)}(s) x_q^{(j)}(s) - \sum_{j=0}^{n-1} \binom{n-1}{j} k_{pq}^{(n-1-j)}(s,s) x_q^{(j)} \right] \\
- \sum_{i=0}^{n-2} \sum_{j=0}^{n-2-i} \binom{n-2-i}{j} \left[ k_{pq}^{(i+1)}(s,s) \right] (n-2-i-j) x_q^{(j)}(s) \\
- \sum_{j=0}^{n} \frac{1}{j!} \left( \int_{0}^{s} k_{pq}^{(n)}(s,t)(t-s)^{j} \, dt x_q^{(j)}(s) \right) = y_p^{(n)}. \] (10)

Here we assume that \( x_q(s) \) approximate to a Taylor polynomial of degree \( n = N \). In this case, we may take \( i, j = 0, 1, \cdots, N \).

Then equation (10) makes a system of \( m(N+1) \) linear equations for the \( m(N + 1) \) unknown coefficients \( x_q^{(0)}(s), x_q^{(1)}(s), x_q^{(2)}(s), \cdots, x_q^{(N)}(s) \), where \( q = 1, 2, \cdots, m \).

These can be solved numerically by standard method. In addition, this system can be put in a matrix forms as
\[
TX + Y = 0. \] (11)

Where \( T, G \) and \( F \) are matrices which are defined by
\[
T = \begin{pmatrix}
T_{00} & T_{01} & \cdots & T_{0,m(N+1)} \\
T_{10} & T_{11} & \cdots & T_{1,m(N+1)} \\
& & \cdots & \\
T_{m(N+1),0} & T_{m(N+1),1} & \cdots & T_{m(N+1),m(N+1)}
\end{pmatrix}
\]

and
\[
G = [x_1(s), x_1'(s), \cdots, x_m^{(N)}(s)]^t,
\]
\[
F = [y_1(s), y_1'(s), \cdots, y_m^{(N)}(s)]^t.
\]

If the determinant \( |T| \neq 0 \), then the matrix equation (11) may be written in the form
\[
G = -T^{-1}F. \] (12)

Thus the coefficients \( x_q^{(j)}(s) \) for \( q = 1, 2, \cdots, m \) and \( j = 0, 1, \cdots, N \) are uniquely determined by equation(12). Therefore IAEs (1) has a unique solution. This solution is given by the Taylor polynomial
\[
x_q(t) = \sum_{j=0}^{N} \frac{1}{j!} x_q^{(j)}(s)(t-s)^{j}, \quad q = 1, 2, \cdots, m. \] (13)

**Keywords:** Integral-algebraic equations, Taylor expansion, Volterra integral equations, numerical method, approximate solutions, error analysis.

**AMS Subject Classification:** 34A34.

**References**

MAGNETIC MOMENT OF ELECTRONS IN DILUTED MAGNETIC SEMICONDUCTOR QUANTUM RING

A.M. BABANLI$^{1,2}$, B.G. IBRAGIMOV$^2$

$^1$Department of Physics, University of Suleyman Demirel, 32260 Isparta, Turkey
$^2$Institute of Physics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan
Faculty of Physics, Baku State University, Baku, Azerbaijan
e-mail: arifbabanli@sdu.edu.tr

ABSTRACT. In the present paper, we have investigated the magnetization of electrons in a diluted magnetic semiconductor (DMS) quantum ring. We take into account the effect of Rashba spin-orbit interaction, the exchange interaction and the Zeeman term on the magnetization. We have calculated the energy spectrum and wave function of the electrons in diluted magnetic semiconductor quantum ring. Moreover, we have calculated the magnetic moment as a function of the magnetic field for strong degenerate electron gas at finite temperature of a diluted magnetic semiconductor quantum ring.

Keywords: Rashba effect, magnetic moment, heat capacity, diluted magnetic semiconductor.

AMS Subject Classification: 35Q41.

1. INTRODUCTION AND MAIN RESULTS

In the last decade enormous attention has been devoted toward control and engineering of spin degree of freedom at mesoscopic scale, usually referred to as spintronics [3]. Important class of materials for spintronics forms diluted magnetic semiconductors (DMS). In a previous paper [1] we calculated the heat capacity and magnetization of a DMS quantum ring for Boltzmann statistics. The aim of this paper is to generalize the theory of free-electron Landau diamagnetism so as to include parabolic of the Fock-Darwin type confinement. In this way we move from classical statistics to the degenerate Fermi limit. We take into account the effects of the Zeeman and exchange terms on the magnetic moment of DMS quantum ring, the electron is assumed to be moving in a parabolic potential of the Fock-Darwin type given by [1]:

\[ V_c(\rho) = \frac{V_0 \rho^2}{2 R^2}, \rho \leq R, \]  

(1)

where \( V_0 \) defines the depth of this potential and \( \rho \) is the distance of electron from the centre of the DMS quantum ring. The quantum ring is subjected to a uniform magnetic field \( \overrightarrow{H} = (0, 0, H) \) normal to the quantum ring plane. We assume that the spin-orbit interaction is described by the Rashba Hamiltonians [1]. The total Hamiltonian of the system is given by:

\[ H = \frac{1}{2 m_n} \left( \overrightarrow{P} + e \overrightarrow{A} \right)^2 + V_c(\rho) + \frac{1}{2} g \sigma_z \mu_B H + \sigma_z \alpha_0 \frac{dV(\rho)}{d\rho} \left( -i \frac{1}{\rho} \frac{\partial}{\partial \phi} + \frac{e H \rho}{2 \hbar} \right) + H_{ex}, \]  

(2)
where $m_n$ is the electron mass, $\sigma_z$ is the Pauli $z$ matrix, $\sigma_0$ is the Rashba spin-orbit coupling parameter, $g$ represents the Lande factor. In the mean field approximation the exchange Hamiltonian term can be written as:

$$H_{ex} = \frac{1}{2} \langle S_z \rangle N_0 x J_{sd} \sigma_z = 3A \sigma_z,$$

(3)

where $J_{sd}$ is a constant which describes the exchange interaction, $N_0$ is the density of the unit cells. For uniform magnetic field, $H$ directed along $z$-axis, the vector potentials in cylindrical polar coordinates have the components $A_\phi = \frac{H \rho}{2}, A_\rho = 0$ and the solution of Schrödinger equation has been known [1]. The electron energy levels given by [1, 2]:

$$E_{nl\sigma} = \hbar \Omega_\sigma \left( n + \frac{1}{2} + \frac{|l|}{2} \right) + \frac{l \cdot h \omega_c}{2} + \frac{1}{2} g \mu_B H + 3A \sigma + \sigma \alpha \cdot \frac{l}{R},$$

(4)

where $\sigma = \pm 1$ and we have used notations:

$$\Omega_\sigma = \sqrt{4 \omega_0^2 + \omega_c^2 + \sigma \alpha \cdot \omega_c \frac{R}{h \cdot \hbar}}, \omega_0 = \sqrt{\frac{V_0}{m_n R^2}}, \omega_c = \frac{eH}{m_n}.$$

(5)

The partition function for the Boltzmann statistics is given by:

$$z = \sum_{\sigma} \frac{1}{2} e^{-\frac{k_BT}{\hbar \omega_c}} = \left[ e^{-\frac{k_BT}{\hbar \omega_c}} \right]_{b_{\sigma}} = \hbar \omega_c \left( 1 + \sigma \frac{2\alpha}{h \omega_c R} \right), d = \frac{1}{2} g \mu_B H + 3A,$$

(6)

where $E_{nl\sigma}$ is the energy spectrum of considered system, $k_B$ is the Boltzmann constant.

To calculate thermodynamic potential $\Omega$ we use an approach based on calculating the classical integrand in (8) contributes significantly:  

$$\Omega = -k_B T \int_{-\infty}^{\infty} \frac{\pi \xi}{\sin(\pi \xi)} \cdot \frac{e^{\frac{k_BT}{\hbar \omega_c}} \xi}{\xi^{2} - z} dz,$$

(7)

where $\mu$ is the chemical potential of the gas. If we change to the dimensionless variable of integration $z = \frac{b_\sigma}{2k_BT} \xi$, Eq. (7) takes the form

$$\Omega_\sigma = \frac{2\pi}{3} \int_{-\infty}^{\infty} \frac{e^{\frac{2k_BT}{b_\sigma}} \xi}{\sin(\pi \xi) \xi^{2} - z} dz,$$

(8)

where $B_\sigma = \frac{\hbar \Omega_\sigma}{b_\sigma}$. The finite temperature effects are represented by an expansion the functions

$$\frac{e^{\frac{2k_BT}{b_\sigma}} \xi}{\sin(\pi \xi) \xi^{2} - z} \approx 1 + \frac{2\pi^2 k_B^2 T^2}{3b_\sigma^2} \xi^2 + \ldots.$$

(9)

For the low fields $\frac{\mu}{k_BT} >> 1$, only the small $z$ behavior of the non-exponential portion of the integrand in (8) contributes significantly:

$$\frac{1}{\cos h (B_\sigma z) - \cosh(z)} \approx \frac{1}{z^2} \frac{2}{B_\sigma^2 - 1} + \frac{1}{6} \frac{B_\sigma^2}{1 - B_\sigma^2} + \ldots.$$

(10)

Inserting (9) and (10) into (8) and using the formula

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dz}{z^{\delta}} dz = \frac{i^{\delta-1} \Gamma(\delta)}{\Gamma(\delta)},$$

(11)
when $\delta$ positive, we find
\[ \Omega = -\frac{1}{4} \sum_{\sigma = \pm 1} b_\sigma \left( \frac{1}{2} \frac{D^2_\sigma}{B^2_\sigma - 1} + \frac{D_\sigma}{6} + \frac{1 + B^2_\sigma}{1 - B^2_\sigma} + \frac{4k_B^2 T^2}{3\hbar^2 (B^2_\sigma - 1)} D_\sigma \right), \]  

(12)

where $D_\sigma = \frac{2\mu - \sigma d}{b_\sigma}$. In the absence of spin degree of freedom the thermodynamic potential
\[ \Omega = -\frac{1}{6} \frac{\mu^3}{\hbar^2 \omega_0^2} + \frac{\mu}{6} \left( 1 + \frac{\omega_0^2}{2\omega_0^2} - \frac{2k_B^2 T^2}{\hbar^2 \omega_0^2} \right). \]

(13)

The magnetic moment of electrons in the quantum ring at the chemical potential $\mu = \text{const}$ is
\[ M = -\frac{1}{A} \left( \frac{\partial \Omega}{\partial H} \right)_\mu = -\frac{2H \mu B^2}{3 \hbar^2 \omega_0^2} \left( \frac{m_0}{m_n} \right)^2, \]

(14)

where $A$ is the area of quantum ring, $m_0$ is the free electron mass. We shall take as the area of the cross-section of the potential well where $\mu = \frac{m_n \omega_0^2 r^2}{2}$ and can be written $A = \frac{2\pi \mu}{m_n \omega_0^2}$.

\[ \frac{M}{\mu_B} = \frac{m_n B^2 H}{3\pi \hbar^2} \left( \frac{m_0}{m_n} \right)^2. \]

(15)

Thus the magnetization is independent of the confinement parameter $\omega_0$.

We next turn to de Haas–van Alphen oscillatory behavior in the magnetization in quantum ring with Rashba spin-orbit coupling. The integrand in (8) has simple poles and the points
\[ 2l\pi/(B_\sigma \pm 1), \quad l = \pm 1, \pm 2, \ldots \]

along the imaginary axis. Evaluating (8) by closing the integration by a large semicircle to the left, and summing the residues we have:
\[ \Omega = k_B T \sum_{\sigma = \pm 1, l = 1} \frac{(-1)^{l+1}}{2l} \left[ \frac{\sin \left( \frac{\mu - \sigma d 4\pi}{B_\sigma + 1} \frac{B_\sigma - 1}{B_\sigma + 1} \pi \right)}{\sin \left( \frac{4\pi}{B_\sigma + 1} \frac{B_\sigma - 1}{B_\sigma + 1} \pi \right)} \right. \]
\[ \left. \frac{\sin \left( \frac{\mu - \sigma d 4\pi}{B_\sigma - 1} \frac{B_\sigma + 1}{B_\sigma - 1} \pi \right)}{\sin \left( \frac{4\pi}{B_\sigma - 1} \frac{B_\sigma + 1}{B_\sigma - 1} \pi \right)} \right]. \]

(16)

The structure of the resulting oscillations is quite complex since $B_\sigma$ is strongly field dependent, and includes spikes where $\frac{B_\sigma - 1}{B_\sigma + 1}$ possesses integer values. At very high fields, so $B_\sigma$ approaches 1 the amplitude the first term in (16) will grow without as expected. Differentiating only the rapidly oscillating factors in Eq. (16) we find the magnetization:
\[ \frac{M}{\mu_B} = \frac{-k_B T m_0}{m_n} \sum_{\sigma = \pm 1, l = 1} \frac{(-1)^{l+1}}{2l} \left[ \frac{\cos \left( \frac{\mu - \sigma d 4\pi}{B_\sigma + 1} \frac{B_\sigma - 1}{B_\sigma + 1} \pi \right)}{\sin \left( \frac{4\pi}{B_\sigma + 1} \frac{B_\sigma - 1}{B_\sigma + 1} \pi \right)} \right. \]
\[ \left. \frac{\cos \left( \frac{\mu - \sigma d 4\pi}{B_\sigma - 1} \frac{B_\sigma + 1}{B_\sigma - 1} \pi \right)}{\sin \left( \frac{4\pi}{B_\sigma - 1} \frac{B_\sigma + 1}{B_\sigma - 1} \pi \right)} \right]. \]

(17)

REFERENCES
ON SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATIONS WITH
INTEGRAL BOUNDARY CONDITIONS

SAMEER BAWANEH

1Department of Mathematics, Eastern Mediterranean University Famagusta, Turkey
e-mail: sameer.b@outlook.com

1. INTRODUCTION

In this paper, we study the following coupled system of Caputo type sequential fractional differential
\[ \begin{align*}
&D^{\alpha - 1}_{\rho, T} (D + k) x(t) = f(t, x(t), y(t)), \quad t \in [0, T], 0 < \alpha \leq 1, k > 0, \\
&D^{\beta - 1}_{\rho, T} (D + k) y(t) = g(t, x(t), y(t)), \quad t \in [0, T], 0 < \beta \leq 1, k > 0,
\end{align*} \]  

Supplemented with integral boundary conditions of the form:
\[ \begin{align*}
&\int_0^T x(s) \, ds = \rho_1 y(\zeta_1), \quad \int_0^T x'(s) \, ds = \rho_2 y'(\zeta_2), \\
&\int_0^T y(s) \, ds = \mu_1 x(\eta_1), \quad \int_0^T y'(s) \, ds = \mu_2 x'(\eta_2),
\end{align*} \]

where $D^{\alpha}_{\rho}, q = \alpha, \beta$, denote the Caputo fractional derivatives of order $q$, respectively, and $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, are given continuous functions, and $k, \rho_1, \rho_2, \mu_1, \mu_2$ are real constants.

The study of coupled systems of fractional-order differential equations is very significant as such systems appear in a variety of problems of applied nature, especially in biosciences. The paper is organized as follows. In Section 2, we recall some basic concepts of fractional calculus and present some auxiliary lemmas. The main results are presented in Section 3. We give two results: the first one concerning the uniqueness of solutions is established by applying Banach’s contraction mapping principle, while the second one dealing with the existence of solutions is derived by applying Leray–Schauder’s alternative.

Let $\psi, \phi \in C([0, T], \mathbb{R})$, then the unique solution of the problem:
\[ \begin{align*}
&D^{\alpha - 1}_{\rho, T} (D + k) x(t) = \psi(t), \quad 1 < \alpha \leq 2, \\
&D^{\beta - 1}_{\rho, T} (D + k) y(t) = \phi(t), \quad 1 < \beta \leq 2, \\
&\int_0^T x(s) \, ds = \rho_1 y(\zeta_1), \quad \int_0^T x'(s) \, ds = \rho_2 y'(\zeta_2), \\
&\int_0^T y(s) \, ds = \mu_1 x(\eta_1), \quad \int_0^T y'(s) \, ds = \mu_2 x'(\eta_2),
\end{align*} \]

is
\[ x(t) = \Delta e^{-kt} \left( \frac{\mu_2}{\Gamma(\alpha - 1)} \int_0^{\eta_2} (\eta_2 - m)^{\alpha - 2} \psi(m) \, dm - k\mu_2 \int_0^{\eta_2} e^{-k(\eta_2 - s)} \int_0^s (s-m)^{\alpha - 2} \psi(m) \, dm \, ds \\
- \int_0^T \int_0^s \frac{\beta - 2}{\Gamma(\beta - 1)} \phi(m) \, dm \, ds + k \int_0^T \int_0^s e^{-k(s-x)} \int_0^x (x-m)^{\beta - 2} \phi(m) \, dm \, dx \, ds \right) \]
\[+ \lambda e^{-kt} \left( \frac{\partial^2}{\partial t^2} \int_0^s (\zeta_2 - m)^{\beta-2} \phi(m) dm + k \rho_2 \int_0^s (\zeta_2 - s)^{\beta-2} \phi(m) dm ds + k f_0^T f_0^s e^{-k(s-x)} \int_0^x (x-m)^{\alpha-2} \psi(m) dm dmds \right)\]

\[+ \int_0^T \frac{1}{T^2 - \mu_1 \rho_1} \left( \frac{\partial^2}{\partial t^2} \int_0^s (\eta_2 - m)^{\alpha-2} \psi(m) dm - \lambda \int_0^T \int_0^s e^{-k(s-x)} \int_0^x (x-m)^{\beta-2} \phi(m) dm dmdx ds \right)\]

Assume Theorem 1.

Here, we have

\[\int_0^T \int_0^s (s-m)^{\alpha-2} \psi(m) dm dmds,\]

\[y(t) = \theta e^{-kt} \left( \int_0^T \int_0^s (\eta_2 - m)^{\alpha-2} \psi(m) dm - \lambda \int_0^T \int_0^s e^{-k(s-x)} \int_0^x (x-m)^{\alpha-2} \phi(m) dm dmdx ds \right)\]

\[+ \tau e^{-kt} \left( \int_0^T \int_0^s (\zeta_2 - m)^{\beta-2} \phi(m) dm + \kappa \rho_2 \int_0^T \int_0^s e^{-k(s-x)} \int_0^x (x-m)^{\beta-2} \phi(m) dm dmdx ds \right)\]

\[+ \int_0^T \frac{1}{T^2 - \mu_1 \rho_1} \left( \int_0^T \int_0^s (s-m)^{\alpha-2} \psi(m) dm + \tau \int_0^T \int_0^s (s-x)^{\alpha-2} \psi(m) dm dmdx ds \right)\]

\[T \mu_1 \int_0^s e^{-k(s-x)} \int_0^x (x-m)^{\alpha-2} \phi(m) dm dmds + \mu_1 \rho_1 \int_0^T e^{-k(s-x)} \int_0^x (x-m)^{\beta-2} \phi(m) dm dmdx ds\]

where

\[A = (\theta T \rho_1 e^{-k(\zeta_1)} + \kappa \rho_1 \Delta e^{-k\eta_1} + (e^{-kT} - 1) (\Delta T + \theta \rho_1))\]

\[B = (\theta T \rho_1 e^{-k(\zeta_1)} + \kappa \rho_1 \Delta e^{-k\eta_1} + (e^{-kT} - 1) (\Delta T + \theta \rho_1))\]

\[C = (\Delta T \mu_1 e^{-k\eta_1} + \kappa \rho_1 \theta e^{-k(\zeta_1)} + (e^{-kT} - 1) (\theta T + \Delta \mu_1))\]

\[D = (\lambda \Delta T \mu_1 e^{-k\eta_1} + \kappa \rho_1 \theta e^{-k(\zeta_1)} + (e^{-kT} - 1) (\theta T + \Delta \mu_1))\]

\[\Delta = \frac{-k \rho_2 e^{-k\zeta_2}}{(e^{-kT} - 1)^2 - k^2 \rho_2 \mu_2 e^{-k(\zeta_2 + \eta_2)}}, \quad \lambda = \frac{e^{-kT} - 1}{(e^{-kT} - 1)^2 - k^2 \rho_2 \mu_2 e^{-k(\zeta_2 + \eta_2)}}, \quad \sigma = \Delta k \mu_2 e^{-k\eta_2} - 1.\]

**Theorem 1.** Assume \(f, g : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}\) are jointly continuous functions and there exist constants \(h_1, h_2, h_3, h_4 \in \mathbb{R}\), such that \(\forall x_1, x_2, y_1, y_2 \in \mathbb{R}, \forall t \in [0, T]\), we have

\[|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq h_1 (|x_1 - x_2| + |y_1 - y_2|),\]

\[|g(t, x_1, x_2) - g(t, y_1, y_2)| \leq h_2 (|x_1 - x_2| + |y_1 - y_2|).\]

If

\[h_1 (N_1 + N_2) + h_2 (N_2 + N_4) < 1\]

then the BVP has a unique solution on \([0, T]\).
AMS Subject Classification: 34A08.

References


AN AUGMENTED LAGRANGIAN RELAXATION BASED SUBGRADIENT APPROACH TO AIRCRAFT MAINTENANCE ROUTING PROBLEM

K. GULNAZ BULBUL¹, REFAIL KASIMBEYLI²

¹Anadolu University, Faculty of Aeronautics and Astronautics, Department of Aviation Management, Eskiehir, Turkey
²Anadolu University, Faculty of Engineering, Department of Industrial Engineerin, Eskiehir, Turkey

e-mail: kgbulbul@anadolu.edu.tr

Airline planning and scheduling is a complex process that generally starts one year ahead of the operation and continues till departure, with necessary modifications. Airline schedule, the main plan of an airline business, is a tool that matches existing resources with demand [1]. This tool, consists of aircraft and crew schedules, aims to maximize the profit of the company by considering strategic and marketing goals [2,3]. Since number of decision variables and constraints and amount of data that should be handled are enormous, it is very hard to handle airline scheduling problem as a whole. Thus, this problem is divided into some main sub-problems as airline scheduling, aircraft scheduling, crew scheduling and irregular operations management.

Aircraft maintenance routing is the third phase in airline planning and scheduling process, subsequent to flight scheduling and fleet assignment phases. The main concern of aircraft maintenance routing problem is to determine a sequence of flight legs for an individual aircraft so that the maintenance requirements, which arise from regulations, are not violated. Minimizing number of aircrafts used, maximizing robustness, maximizing passenger-attractive connections and maximizing through value are some main objectives that are considered in previous studies.

In airline industry there are three main business processes that shapes the definition of the aircraft maintenance routing problem; strings, big cyle and one-day routes. Equal aircraft utilization can be an important concern for airlines. In big cycle approach of the flights of a specific fleet are assigned in a cyclic manner in order to ensure equal aircraft utilization. All aircrafts are assigned to the cycle constructed with a one-day delay between each [4].

This study covers the aircraft maintenance routing problem as a big-cycle, ensuring equal aircraft utilization while maximizing through value. Problem is modeled as an asymmetric travelling salesman problem with side constraints. Model is as follows:

Sets:
- $I = J$: flight legs
- $B$: \{i ∈ I, j ∈ J : h(i) = t(j), i ≠ j\}
- $S$: Arcs (flight legs) in a subtour
- $P$: Principal violation paths
- $P'$: set of arcs of $P$\ last arc

Parameters:
- $v_{ij}$: if flight i is followed by flight j
- $f(j)$: $(j+1)$th arc for $j ∈ P$

Decision Variable:
\[ x_{ij} = \begin{cases} 
1 & \text{if flight } i \text{ is followed by flight } j \\
0 & \text{otherwise.} 
\end{cases} \]

**Constraints:**

\[
\sum_{j \in B} x_{ij} = 1, \quad \forall I, 
\]  
(1)

\[
\sum_{i \in B} x_{ij} = 1, \quad \forall J, 
\]  
(2)

\[
\sum_{i \in S; j \in I \setminus S; i,j \in B} x_{ij} \geq 1, \quad \forall S \subseteq I, 2 \leq |S| \leq |I| - 2, 
\]  
(3)

\[
\sum_{i \in P'; j \in f(i), i,j \in B} x_{ij} \geq 1 \quad \forall P. 
\]  
(4)

**Objective Function:**

\[
\max \quad z = \sum_{i \in B} v_{ij} x_{ij}. 
\]  
(5)

Constraints (1) and (2) are connectivity constraints that ensures that for every arc \( j \in J \) there is only one connection \( i \in I \) and vice versa. Constraint (3) is continuity break constraint which ensures resulting route to be a cycle, not broken rotations. It is basically the subtour elimination constraint of TSP. Constraint (4) is for eliminating service violation paths by breaking at least one connection in a principal violation path. The objective of the model is to maximize the throughput value.

This study uses one of the main approaches, Lagrange Relaxation, to aircraft maintenance routing problem. Since this problem is an integer, NP-Hard, nonconvex optimization problem conventional subgradient algorithm is not as efficient as desired.

The first generic algorithm based on the sharp augmented Lagrangian, called the Modified Subgradient Algorithm, was introduced by Kasimbeyli in [5]. This algorithm does not require convexity and differentiability of the objective and constraint functions. To the best of our knowledge this is the first augmented Lagrangian-based algorithm which does not use a penalty parameter. It was shown that the algorithm generates strongly monotonically increasing sequence of dual values which converges to the optimal dual value. This was proved under the assumption that there is an efficient unconstrained global optimization method which is used at every iteration to minimize the sharp augmented Lagrangian for the given values of dual variables. The use of conventional global optimization algorithms can make any method based on the sharp Lagrangian function highly time-consuming even for small sized problems.

This assumption was weakened in the modified version of the algorithm, developed by Kasimbeyli, Ustun and Rubinov in [6]. They proposed special update formulas for dual variables in the algorithm, and proved the convergence for the case where the computation of a global minimum of the augmented Lagrangian at every iteration, is not required. Thus, in this modification of the method any local minimization algorithm can be applied for unconstrained minimization of the augmented Lagrangian function. In this paper we utilize this method, modified subgradient algorithm based on feasible values (F-MSG), and use special algorithm to minimize the unconstrained sharp augmented Lagrangian problem at every iteration. The performance of the method is tested on sample examples. Computational results obtained for test problems will be explained in the extended version of this work. F-MSG is as follows [6]:

\[ f_0(x), f(x) : \text{objective function, constraint functions} \]

\[ n : \text{number of iterations} \]

\[ k : \text{number of updates of dual variables} \]

\[ u_k, c_k : \text{dual variables in the } k^{th} \text{ update} \]
Find an element \( x \in \mathbb{R}^n \) and go to step 2.

\[ n_{\epsilon 130} \]

K. GULNAZ BULBUL, REFAIL KASIMBEYLI

\[ u \in \text{an approximate primal solution and } (u, c) \text{ is an approximate dual solution}; \text{ otherwise set } \]

\[ p \in \text{number of cases where a solution to the constraint satisfaction problem can not be found} \]

Step 1: Choose positive numbers \( \epsilon_1, \epsilon_2, \delta_0 \) and \( M \), and any number for \( H_1 \).

Set \( n = 1, p = 0, q = 0 \).

Step 2: Choose \( (u_1, c_1) \in \mathbb{R}^n \times \mathbb{R}^p \) and \( q(1) > 0 \) and set \( k = 1, u_k = u_1, c_k = c_1 \).

Step 3: Given \( (u_k, c_k) \) solve the following constraint satisfaction problem \( P(H_n) \):

Find an element \( x \in S \) such that \( f_0(x) + c_k \parallel f(x) \parallel - \langle u_k, f(x) \rangle \leq H_n \) \( (6) \)

If a solution to (6) does not exist (e.g., if \( l(k) > m \)) then go to Step 6, otherwise if a solution \( x_k \) exists then check whether \( f(x_k) = 0 \) or not. If \( f(x_k) = 0 \) (or if \( \parallel f(x_k) \parallel \leq \epsilon_1 \)) then go to Step 5, otherwise go to Step 4.

Step 4: Update dual variables as

\[ u_{k+1} := u_k - \alpha s_k f(x_k), \]

\[ c_{k+1} := c_k + (1 + \alpha)s_k \parallel f(x_k) \parallel, \]

where \( s_k \) is a positive stepsize parameter defined as

\[ \delta \alpha(H_n - L(x_k, u_k, c_k)) \]

\[ \parallel f(x_k) \parallel^2 \]

with \( \alpha > 0 \) and \( 0 < \delta < 2 \). Also stepsize \( s_k \) corresponding to the dual variables \( (u_k, c_k) \) to satisfy the following property.

\[ s_k \parallel f(x_k) \parallel + c_k - \parallel u_k \parallel > (l)k \]

Set \( k = k + 1 \), update \( l(k) \) in such a way that \( l(k) \to +\infty \), and repeat Step 3.

Step 5: Let \( x_k \) be a solution to (6) with \( f(x_k) = 0 \). In this case \( L(x_k, u_k, c_k) = f_0(x_k) \). Set \( q = q + 1 \) and check \( p \). If \( p = 0 \) then set \( \delta_{n+1} = \delta_n \), otherwise set \( \delta_{n+1} = \frac{1}{2} \Delta_n \). Check \( \Delta_{n+1} \). If \( \Delta_{n+1} < \epsilon_2 \) the stop, \( f_0(x_k) \) is an approximate optimal value, \( x_k \) is an approximate primal solution and \( (u_k, c_k) \) is an approximate dual solution; otherwise set \( H_{n+1} = \min \{ f_0(x_k), H_n - \Delta_{n+1} \} \); \( n = n + 1 \) and go to step 2.

Step 6: Set \( p = p + 1 \) and check \( p \). If \( q = 0 \) then set \( \delta_{n+1} = \delta_n \), otherwise set \( \delta_{n+1} = \frac{1}{2} \Delta_n \). Check \( \Delta_{n+1} \). If \( \Delta_{n+1} < \epsilon_2 \) the stop, \( f_0(x_k) \) is an approximate optimal value, \( x_k \) is an approximate primal solution and \( (u_k, c_k) \) is an approximate dual solution; otherwise set \( H_{n+1} = \min \{ f_0(x_k), H_n + \Delta_{n+1} \} \); \( n = n + 1 \) and go to step 2.

Keywords: Augmented Lagrangian relaxation, subgradient, aircraft.

AMS Subject Classification: 90B50, 90B80, 90C27.

REFERENCES

INVESTIGATION OF A DISCRETE DIRAC SYSTEM WITH AN INTERACTION POINT

ŞERİFENUR CEBESOY\textsuperscript{1}, ELGİZ BAİRİMOV\textsuperscript{2}, ŞEYDA SOLMAZ\textsuperscript{2}

\textsuperscript{1}Çankırı Karatekin University, Turkey
\textsuperscript{2}Ankara University, Turkey
e-mail: scebesoy@karatekin.edu.tr

1. Introduction and preliminaries

Theory of impulsive differential equations dates back to the end of last century [3, 6, 13] and the solutions of some problems for impulsive equations led to a significant improvement over the existing literature. In the recent years, most of the researchers were interested in such problems with impulsive effects for its wide applications in physics, engineering, quantum mechanics and spectral theory. For Sturm–Liouville operator and its discrete analogue, there are a couple of equivalent descriptions used by authors that disturb the continuity such as point interactions, impulsive conditions, transmission conditions, and interface conditions [8–10, 15, 16].

Before presenting the main results, we recall the concepts related with our study.

Eigenvalues and spectral singularities are two parts of continuous spectrum of a nonselfadjoint Sturm–Liouville operator which has been investigated with the pioneering study of Naimark [11]. Spectral singularities were discovered by Naimark and subsequently studied by others [12, 14]. Since then it was understood that spectral singularities cause obstructions for the completeness of the eigenfunctions and they have some physical meanings. Marchenko investigated the Sturm–Liouville boundary value problem [7]

\begin{equation}
y'' + q(x)y = \lambda^2 y, \quad -\infty < x < \infty,
\end{equation}

where \( \lambda \) is a spectral parameter, \( q \) is a complex valued function. (1) admits a pair of solutions fulfilling the conditions

\begin{equation}
\lim_{x \to \pm \infty} e(x, \lambda)e^{\mp i\lambda x} = 1, \quad \lambda \in \mathbb{C}_+ := \{ \lambda \in \mathbb{C}, \text{Im} \lambda \geq 0 \}.
\end{equation}

These bounded solutions are called Jost solutions of (1), which play an important role in the solutions of direct and inverse problems of spectral theory.

A spectral singularity is a point of the continuous spectrum of \( H \) satisfying the eigenvalue equation

\begin{equation}
Hy = \lambda^2 y
\end{equation}

such that the Jost solutions are linearly dependent at these points, in other words, they have a vanishing Wronskian. Note that, the Wronskian of any two solutions \( y, z \) of (1) is defined as

\[ W[y, z] := yz' - y'z. \]

In the present paper, we consider the following discrete Dirac system

\begin{equation}
\begin{cases}
y^{(2)}_{n+1} - y^{(2)}_n = \lambda y^{(1)}_n, \\
y^{(1)}_n + y^{(1)}_{n-1} = \lambda y^{(2)}_n, \\
\end{cases} \quad n \in \mathbb{Z}\setminus\{-1, 0, 1\}
\end{equation}

\( \lambda \).
and the point interaction
\begin{equation}
    y_1(z) = B y_{-1}(z), \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C},
\end{equation}
where
\begin{equation}
    \lambda := 2 \sin \left( \frac{z}{2} \right)
\end{equation}
is a complex spectral parameter. (5) is an impulsive condition for (4) at the point \( n = 0 \) and the matrix \( B \) is used to continue the solution of (4) from negative integer numbers to the positive integer numbers.

Some useful results were obtained for this system without impulsive effects [1, 2, 4, 5]. This paper aims to investigate the eigenvalues and spectral singularities of the impulsive discrete Dirac operator corresponding (4)-(5) boundary value problem depending on the choice of the constants \( a, b, c, d \) and to examine some special cases when the point interaction has \( \mathcal{P}, \mathcal{T}, \) and \( \mathcal{PT} \)-symmetry.

2. Corresponding operator

Let \( T \) be the operator in the Hilbert space \( \ell_2(\mathbb{Z}, \mathbb{C}^2) \) created by the expression (4) and the point interaction (5).

Now, let \( z \in \mathbb{C} \setminus \mathbb{C}_1 \), where \( \mathbb{C}_1 := \{ z : z = (2n + 1)\pi, n \in \mathbb{Z} \} \). Then it is easy to verify that
\begin{equation}
    \varphi_n(z) = \begin{pmatrix} \varphi_n^{(1)}(z) \\ \varphi_n^{(2)}(z) \end{pmatrix} = \begin{pmatrix} e^{iz} \\ -i \end{pmatrix} e^{inz}
\end{equation}
and
\begin{equation}
    \psi_n(z) = \begin{pmatrix} \psi_n^{(1)}(z) \\ \psi_n^{(2)}(z) \end{pmatrix} = \begin{pmatrix} -i \\ e^{iz} \end{pmatrix} e^{-inz}
\end{equation}
are two linearly independent solutions of (4), since the Wronskian of these two functions is defined as
\begin{equation}
    W[\varphi, \psi] := \varphi_n^{(1)}(z)\psi_n^{(2)}(z) - \varphi_n^{(2)}(z)\psi_n^{(1)}(z) = e^{iz} + 1.
\end{equation}

Thus, we can express the general solution \( y_n = \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix} \) of (4) by
\begin{equation}
    y_n(z) = A \varphi_n(z) + B \psi_n(z), \quad n \in \mathbb{Z}^- \cup \mathbb{Z}^+
\end{equation}
and clearly we get
\begin{equation}
    y_n^{(1)}(z) = \begin{cases} A_- e^{iz(\frac{1}{2} + n)} - iB_- e^{-inz}, & n \in \mathbb{Z}^- \\ A_+ e^{iz(\frac{1}{2} + n)} - iB_+ e^{-inz}, & n \in \mathbb{Z}^+ \end{cases}, \quad n \rightarrow \infty
\end{equation}
\begin{equation}
    y_n^{(2)}(z) = \begin{cases} -iA_- e^{inz} + B_- e^{iz(\frac{1}{2} - n)}, & n \in \mathbb{Z}^- \\ -iA_+ e^{inz} + B_+ e^{iz(\frac{1}{2} - n)}, & n \in \mathbb{Z}^+ \end{cases}, \quad n \rightarrow \infty
\end{equation}
where \( A, B, A_\pm \) and \( B_\pm \) are constant coefficients. Moreover, there exist two solutions \( e_n^\pm(z) \) of the equation (4) with the following asymptotical behaviors:
\begin{equation}
    e_n^+(z) \rightarrow \varphi_n(z), \quad n \rightarrow \infty
\end{equation}
and
\begin{equation}
    e_n^-(z) \rightarrow \psi_n(z), \quad n \rightarrow -\infty.
\end{equation}
These are celebrated as Jost solutions of (4) and they are both analytic with respect to \( z \) in \( \mathbb{C}_+:\{z:z \in \mathbb{C}, \text{Im} \, \lambda > 0\} \) and continuous up to the real axis. According to the point interaction (5), we write

\[
\begin{pmatrix} y_1^{(1)}(z) \\ y_1^{(2)}(z) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_{-1}^{(1)}(z) \\ y_{-1}^{(2)}(z) \end{pmatrix}
\]

(10)

and finally (10) turns into

\[
\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = \mathbf{M} \begin{pmatrix} A_- \\ B_- \end{pmatrix},
\]

(11)

where

\[
\mathbf{M} := \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} := N_1^{-1} \mathbf{B} N_2
\]

(12)

with

\[
N_1 := \begin{pmatrix} e^{\frac{iz}{2}} & -ie^{-iz} \\ -ie^{iz} & e^{\frac{iz}{2}} \end{pmatrix}, \quad N_2 := \begin{pmatrix} e^{-\frac{iz}{2}} & -ie^{iz} \\ -ie^{-iz} & e^{-\frac{iz}{2}} \end{pmatrix}.
\]

**Keywords:** Impulsive conditions, Dirac systems, eigenvalues, spectral singularities, symmetries.

**AMS Subject Classification:** 34B37, 34L05, 39A13, 81Q05.

**References**


COVARIANCE SWITCHING IN VIBRATION CONTROL

PATRIZIO COLANERI

1 DEIB - Politecnico di Milano and IEIIT-CNR, Piazza Leonardo da Vinci 32, 20133 Milano, Italy
E-mail: patrizio.colaneri@polimi.it

1. INTRODUCTION

The problem of minimization of vibrations in mechanical systems and civil structures is usually tackled through the use of Tuneable Vibration Absorbers (TVA), [1,4]. These devices comprise a seismic mass mounted on a spring-damper suspension. In order to control the response over a wide frequency band, a rather large number of TVA devices should be employed, each tuned to a specific structural mode that resonates within the targeted frequency band [3]. However, recently Zilletti et al. [5] have proposed a covariance switching approach for TVAs, such that it can control the resonant responses due to the multiple modes of a structure. The analysis deeply benefits from the theory of time-varying and the theory of linear switched systems extended to cope with stochastic white noise input and energy bounded input. To this end a switching covariance strategy is considered for energy attenuation. The results are nontrivial extension of previous works on so-called Lyapunov-Metzler ”argmin” switching operation, see [2]. The present study refers to a practical system formed by a thin walled circular duct structure on which an electro-mechanical TVA was mounted. The TVA is formed by a coil magnet transducer, with the magnet block mass mounted on an elastic suspension. The coil-magnet transducer is driven by an open loop control system to switch the stiffness and damping coefficients of the elastic suspension so that the TVA fundamental resonance frequency is varied and the TVA damping ratio is kept constant.

2. COVARIANCE SWITCHING

The system under consideration has the form

\[ \dot{x} = A_{\sigma(t)}x + Bu, \]
\[ z = Cx, \]

where \( \sigma(t) \) is a switching signal taking values in the set \( \mathcal{M} = \{1, 2, \cdots, M\} \) and \( u \) and \( z \) are scalar variables. When \( x(0) = 0 \) and \( u(t) \) is a white noise with identity intensity, the covariance matrix \( P(t) = E[x(t)x(t)^\top] \) satisfies the differential Lyapunov equation

\[ \dot{P}(t) = A_{\sigma(t)}P(t) + P(t)A_{\sigma(t)}^\top + BB^\top, \quad P(0) = 0, \tag{1} \]

with initial condition \( P(0) = 0 \).

A formidable optimal control problem is to find \( \sigma(t) = f(P(t)) \) such as to maintain \( P(t) \) bounded and minimizing the energy cost. Finding the optimal switching law is a challenging task. A suboptimal procedure is based on the so-called Lyapunov Metzler inequalities, see [2] for a pioneering paper on this procedure, i.e.
\[ A_i^T X_i + X_i A_i + \sum_{j=1, j \neq i}^{M} \lambda_{ij} (X_j - X_i) + C^T C < 0, \quad \lambda_{ij} \geq 0, \quad i \neq j = 1, 2, \ldots, M, \tag{2} \]

where \( \lambda_{ij} \geq 0 \) are tuning parameters, along with the switching law

\[
\sigma(t) = \arg \min_{j \in I(P)} \text{trace}[(A_{\sigma(t)} P(t) + P(t) A_{\sigma(t)} + B B^T) X_j],
\tag{3}
\]

where \( I(P) \) is the set of minimizers, i.e. \( I(P) = \{ i : V(P) = \text{trace}(P X_i) \} \).

**Theorem 1.** Consider the covariance equation (1) and the solutions \( X_i \succ 0 \) of Lyapunov Metzler equations (2) for some fixed parameters \( \lambda_{ij} \geq 0, \quad i \neq j \). The switching control law (3) ensures boundedness of \( P(t), \quad t \geq 0 \) and

\[
J_2 = \lim_{T \to \infty} \frac{1}{T} \int_0^T \text{trace}(CP(t)C^T) dt < \lim_{T \to \infty} \frac{1}{T} \int_0^T \text{trace}[B^T X_{\sigma(t)} B] dt.
\tag{4}
\]

**Remark 1.** The parameters \( \lambda_{ij} > 0 \) can be optimized in order to minimize the upper bound. Notice that if one takes \( \lambda_{ij} = 0 \) (the solutions \( X_i \) always exist if \( A_i \) are Hurwitz stable matrices) it follows that

\[
J_2 \leq \sum_{k=1}^{M} \alpha_k \| G_k(s) \|^2_2,
\]

where \( \alpha_k \) is the fraction of time (average dwell time) spent in the \( k \)-th mode \( \sigma(t) = k \) and \( G_k(s) = C(sI - A_k)^{-1} B \). The guaranteed upper bound is greater than the minimum of the norms \( \| G_k(s) \|^2_2, \quad k = 1, 2, \ldots, M \). With this choice the upper bound does not improve on the minimum between the norms associated with the time-invariant modes.

### 3. Model Problem

The practical model problem is composed by a thin walled circular duct flexible structure equipped with an electro-mechanical TVA. Two feedback loops produce a pair of reactive forces proportional to the relative displacement and relative velocity between the casing and the suspended mass of the TVA, which is equivalent to producing elastic and damping reactive forces in parallel with the TVA suspension component. As a result, the stiffness and damping properties of the TVA suspension can be varied by changing the feedback gains of the two feedback loops. The physical properties and dimensions of the duct and TVA system are summarised in Tables 1 and 2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>length</td>
<td>( L = 1500 \text{ mm} )</td>
</tr>
<tr>
<td>diameter</td>
<td>( D = 200 \text{ mm} )</td>
</tr>
<tr>
<td>thickness</td>
<td>( h = 1 \text{ mm} )</td>
</tr>
<tr>
<td>density</td>
<td>( \rho_c = 8050 \text{ Kg/m}^3 )</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>( E_c = 200 \times 10^9 \text{ N/m}^2 )</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>( \nu_c = 0.33 )</td>
</tr>
<tr>
<td>Modal damping ratio</td>
<td>( \zeta_c = 0.01 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Seismic mass</td>
<td>( m = 67 \text{ g} )</td>
</tr>
<tr>
<td>Suspension stiffness</td>
<td>( k = 2152 \text{ N/m} )</td>
</tr>
<tr>
<td>Suspension damping ratio</td>
<td>( \zeta = 4% )</td>
</tr>
</tbody>
</table>
Figure 1. Power spectra of $z(t)$ without TVA (black line) and with the Lyapunov-Metzler switching TVA (red line).

The spectral response of the system is analysed in the frequency range between 50 and 350 Hz. In this range, the flexural response of the duct wall is dominated by the resonant responses of three flexural modes. The dynamic response of the system has therefore been modeled with the canonical state-space formulation:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad z(t) = Cx(t),$$

where $u(t)$ represents the primary excitation force, $x(t)$ is the $(8 \times 1)$ state-vector

$$x = \begin{bmatrix} q_1 & q_2 & q_3 & w & \dot{q}_1 & \dot{q}_2 & \dot{q}_3 & \dot{w} \end{bmatrix}^T,$$

which refers to the generalised coordinates (modal coordinates) $q_1(t), q_2(t), q_3(t)$ for the responses of the first three flexural modes of the duct structure and the physical coordinate $w(t)$ for the response of the TVA seismic mass. Finally $z(t)$ is transverse velocity at the TVA location.

The TVA mounted on the duct structure can be set to control the resonant response due to one of the three flexural modes that characterise its response between 50 and 350 Hz. The idea is that of varying in a discrete manner, the tuning parameters, i.e. the TVA suspension stiffness and damping, in such a way as the resonant responses of all three flexural modes that control the dynamic response of the duct between 50 and 350 Hz are controlled.

Letting $G_i(s) = C(sI - A_i)^{-1}B$ be the transfer functions of the system associated with a TVA tuned at flexural mode $i$, $i = 1, 2, 3$, it is easy to compute the minimum $H_2$ norm, resulting in $\min_{i=1,2,3} \|G(s)\|_2^2 = 3.41$. For such modes the best covariance switching law presented in Theorem 2 has been computed over hundreds values of the tuning parameters $\lambda_{ij}$. The result is the Isce- periodic switching law characterized in a period by the residence intervals: $A_1$ in the interval $[0,0.018]$, $A_3$ in the interval $[0.019,0.039]$, $A_2$ in the interval $[0.04,0.063]$ and $A_3$ in the interval $[0.064,0.099]$. The resulting cost is 2.255. The reduction effect is apparent from Fig.1.

**Keywords:** Switched systems, switching control, stochastic systems.

**AMS Subject Classification:** 93C05, 93C30, 93C80.

**REFERENCES**

RADIAL POSITION-MOMENTUM UNCERTAINTIES FOR THE INFINITE SPHERICAL WELL AND THE FISHER ENTROPY

QIAN DONG\textsuperscript{1}, ARIADNA J. TORRES-ARENAS\textsuperscript{1}, GUO-HUA SUN\textsuperscript{2}, SHI-HAI DONG\textsuperscript{1}

\textsuperscript{1}Laboratorio de Información Cuántica, Instituto Politécnico Nacional, Mexico
\textsuperscript{2}Catedrática CONACyT, Centro de Investigación en Computación, Instituto Politécnico Nacional, Mexico
e-mail: dongsh2@yahoo.com

Based on previous work \cite{1-5}, we present the analytical expression of the product of the radial position and momentum uncertainties $\Delta r$ and $\Delta p_r$ for the infinite spherical well. We find a few interesting features. First, the uncertainty $\Delta r$ increases with the radius $R$ and the quantum number $n$, the $n$-th root of the spherical Bessel function, but finally it will arrive to a constant for a large $n$ and it decreases with the angular momentum number $l$. It is seen that the $\Delta r$ becomes imaginary, which arises from the fact that the moving particle is shifted to axis $y$ suddenly from the original axis $x$ as the quantum number $n$ increases. Also, the $\Delta r$ becomes zero when the $n$ increases for a given $l$. This means that the particle is spiraling around a circle whose radius $r < R$ changes between a varying radius and a constant but with an increasing radius $r$ as the $n$ increases. Finally, the particle will moving around a circle with a maximum radius $R$. Second, the relative dispersion $\Delta r/(\langle r \rangle)$ is independent of the radius $R$ and it increases with the quantum number $n$ but decreases with the quantum number $l$. Third, the radial momentum uncertainty $\Delta p_r$ decreases with the radius $R$ and increases with the quantum numbers $l > 0$ and $n$. We notice that there exists a turning point for the uncertainty $\Delta p_r$ when $l = 1$ and $n > 1$. This also leads to the similar problem to the product $\Delta r \Delta p_r$. Fourth, the product $\Delta r \Delta p_r$ is independent of the radius $R$ and increases with the quantum numbers $l > 0$ and $n$. Finally, we obtain the analytical expression of the Fisher entropy and notice that it decreases with the radius $R$ but increases with the quantum numbers $l$ and $n$. Also, we find that the Cramer-Rao uncertainty relation is satisfied and it increases with the quantum numbers $l$ and $n$, respectively. Some important results are given below.

Figure 1. Radial wave functions for the states $l = 0, n = 1$, $l = 1, n = 2$ and $l = 2, n = 3$, respectively.
Figure 2. Variation of the uncertainty $\Delta r$ with respect to quantum number $l$ with values $l = 12, 30, 50, 85$.

Figure 3. The relative dispersion $\triangle r/\langle r \rangle$ is independent of the radius $R$.

Figure 4. The uncertainty $\Delta r \Delta \rho_r$ is independent of the radius $R$. 
Figure 5. Variation of the uncertainty $\Delta r \Delta p_r$ with respect to quantum numbers $l$ and $n$.

Figure 6. Variation of the Fisher entropy with respect to the radius $R$. We find that it decreases with the $R$.

**Keywords:** Position and momentum uncertainty relation, infinite spherical well, fisher entropy, confined system.

**AMS Subject Classification:** 81Q05, 35Qxx, 42A38.

**References**

CONTROLLED QUEUES WITH CORRELATED ARRIVAL PROCESS AND RESERVE SERVERS FOR SOLVING ENERGY SAVING PROBLEMS IN CLOUD COMPUTING SYSTEMS

A.N. DUDIN\(^1\), S.A. DUDIN\(^1\), V.I. KLIMENOK\(^1\) O.S. DUDINA\(^1\)

\(^1\)Belarusian State University, Minsk, Belarus
\(^2\)Peoples Friendship University of Russia (RUDN University), Moscow, Russia
e-mail: dudin-alexander@mail.ru, dudin85@mail.ru, vklimenok@yandex.ru, dudina@bsu.by

1. Introduction

The important problems related to power consumption and energy saving in many real-world systems, in particular in data centers (server farms), can be successfully solved by means of reservation and subsequent adaptive connection and disconnection of available facilities (servers). Owing to the stochastic nature of arriving flows and processes, which describe the processing of information, mathematical modeling of systems with reserve servers within queueing theory is of current interest. The necessity of adaptive connection (switching on) and disconnection (switching off) of service facilities is explained by the known fact (see, e.g., [2, 6]) that in many real data centers, the servers that are not working, but are not switched off, still consume about 65% of the maximum power consumption of the working servers. Therefore, the topic of analysis of systems with switching on and off additional servers is very important and popular now.

Important feature of modern systems is that the arrival flows are bursty and correlated. Therefore the models with the stationary Poisson arrival process are not appropriate for the analysis. Instead, the queueing systems with the Markovian Arrival Process (MAP) have to be applied. For definition of the MAP and survey of the existing literature, see [1, 5, 7]. A quite restrictive assumption, which is usually made for the multi-server queues, is the assumption that the service time has an exponential distribution. In this abstract we briefly describe results obtained for two multi-server queues with the MAP, reserve servers and the so-called phase-type (PH) distribution of the service time. This distribution is much more general than the exponential distribution and can be used for approximation of an arbitrary distribution. The set of the papers devoted to analysis of queues with reserved servers, MAP arrival process and PH distribution of the service time is very small. In this abstract, we describe in brief two such papers.

In section 2, we refer to the model in which the main and the reserve servers are identical and the latter servers are activated of deactivated depending of the current queue length. The main motivation of the control by the number of active servers is to reduce, under certain economical restrictions on the number of servers in use, the average waiting time of an arbitrary customer. The fate of the customer after its service beginning is not monitored. It is assumed that after a random amount of time defined by the service time distribution, the customer leaves the system. However, if the variance of the service time distribution is large, this customer may spend quite a long time in service. In section 3, we refer to another model in which the main and the reserve servers may be different and the reserve server is activated only if duration of the service of a customer by the main server is too long. The goal of such a type of control is to reduce, again
under economical restrictions on the number of the main and backup servers in use, the average sojourn time of an arbitrary customer. An additional reserve (backup) server is switched on, irrespective of the number of customers in the system, when the duration of service of some customer exceeds some amount of time and is switched off only after completion of the service of this customer.

2. System with identical servers and hysteresis control by the number of the active servers

In [3], a quite complicated queueing model is analysed. In brief description of this model given below, we omit that the customers can be heterogeneous and the service time distribution depends on the type of a customer. It is assumed that the usage of servers is not free, server usage cost is incurred. So, it is necessary to consider not only performance aspects but also the resulting cost/performance ratio. The total number of the identical servers is $N$. Among them, $R$ servers are available (active) all the time, $R < N$, while the other $N - R$ servers can be individually switched on or off depending on the number of customers in the system.

The most popular in the literature strategy of control is the multi-threshold control. In [3], more general hysteresis strategy of switching the servers on and off is applied. Such type of control allows to reduce the number of switchings of the server what is important because in many real-world systems the switching on or off the server may require an essential amount of energy and may be time consuming. The hysteresis strategy is defined as follows. Let the thresholds $K_l$ and $M_l$, $1 \leq l \leq N - R$, be fixed and

$$M_l \leq K_1 < M_2 < \cdots < M_{N-R} \leq K_{N-R}.$$ 

If a customer arrives when $l - 1$ additional servers are in use and the number of customers in the system becomes larger than the threshold $K_l$, the $l$-th additional server is switched on and starts working. If service is completed when $l$ additional servers are in use and the number of customers in the system becomes equal to the threshold $M_l$, the $l$-th additional server is switched off, $1 \leq l \leq N - R$. The multi-threshold strategy of control is the particular case of the hysteresis strategy for which $K_l = M_l$ for all $1 \leq l \leq N - R$.

Under the fixed values of the thresholds $K_l$ and $M_l$, $1 \leq l \leq N - R$, behavior of the system is described by the multi-dimensional Markov chain. In [3], the relevant literature is surveyed in brief, the stationary behavior of the chain is analysed, the key performance measures of the system are obtained and the value of economical cost criterion under any fixed thresholds is computed. Numerical results show high effectiveness of the hysteresis control and importance of account of correlation in the arrival process and variance of service times when making a decision about the control strategy.

3. System with backup servers

In [4], another mechanism of servers activation and deactivation is considered. It is assumed that the servers may be heterogeneous and are divided into two groups: main and backup servers. The number of backup servers does not exceed the number of main servers. Initially, service of a customer is started by an available main server. Simultaneously with starting the service of a customer a timer is switched on. This timer counts a random time having $PH$ distribution. If during this time the service is not completed, an available backup server joins the service (provides help to the main server) of the customer. After the service completion, both servers return to the pools of available main and backup servers, correspondingly. In [4], the relevant literature is surveyed in brief. The behavior of the system is described by the multi-dimensional continuous-time Markov chain. The stationary behavior of the chain is analysed. In particular, the nontrivial ergodicity condition has been derived, the stationary distribution of the system states and the sojourn time (in the case of the lack of deficit of backup servers) have
been found. The main performance measures of the system have been computed. Numerical results have been presented. The results illustrate the feasibility of the proposed algorithms, the effect of correlation in the arrival process, the possibility of the optimal choice of the number of backup servers, and the mean service time after which the assistance of a backup server should be required.

4. Conclusion

Two different mechanisms for providing satisfactory service to customers under the minimal consumption of the energy are described in brief for the multi-server queues with MAP arrival process and PH distribution of the service time. The first mechanism suggests activation of the additional reserve servers when the queue length increases and deactivation when the queue length decreases. The second mechanism does not account the queue length, but monitors the expired service time of each customer in service. When this time exceeds the chosen level, the available (if any) reserve server joins the service of the customer to intensify the service. Both mechanisms sound reasonable for application to optimization of performance measures of the real-world systems with the use of the minimal number of the servers (what leads to decreasing power consumption by the servers). The results can be used for optimization of operation of various cloud computing systems.

5. Acknowledgments

The publication has been prepared with the support of the RUDN University Program 5-100 and Belarusian Republican Foundation for Fundamental Research.

Keywords: Power consumption, queueing system, control by the number of active servers, optimization, Markov arrival process, phase type distribution.

AMS Subject Classification: 60J27, 60J28, 60K25.

REFERENCES

SENSITIVITY ANALYSIS OF ONE OPTIMIZATION PROBLEM WITH PREHISTORY

PHRIDON DVALISHVILI

1 Department of Computer Sciences, Tbilisi, Ivane Javakhishvili Tbilisi State University, Georgia
e-mail: pridon.dvalishvili@tsu.ge

As is known the real processes contain an information about their behavior in the past and are described by differential equations with prehistory [1,2]. In the present work for an optimal control problem with the concentrated and distributed variable delays sensitivity of the minimum of functional with respect to perturbations is investigated.

Let $I = [t_0, t_1]$ be a finite interval and let $O \subset \mathbb{R}^n$ be an open set; suppose that $\Omega$ is the set of measurable control functions $u(t) \in U$, $t \in I$, where $U \subset \mathbb{R}^r$ is a compact set and $D$ is the set of continuously differentiable scalar functions $\tau(t)$, $t \in [t_0, \infty)$ satisfying the conditions: $\tau(t) < t$, $\tau(t) > 0$, $\inf \{\tau(t_0) : \tau \in D\} := \hat{\tau} > -\infty$.

Let $I_1 = [\hat{\tau}, t_1]$ and let $C(I_1)$ be the set of continuous functions $\varphi(t) \in \mathbb{R}^n$, $t \in [\hat{\tau}, t_1]$, equipped with the norm $\|\varphi\|_{I_1} = \sup\{\|\varphi(t)\| : t \in I_1\}$; $\Phi = \{\varphi(\cdot) \in C(I_1) : \varphi(t) \in O, t \in I_1\}$ is the set of initial functions.

To each control $u(\cdot) \in \Omega$ we put into correspondence the controlled differential equation with prehistory (concentrated and distributed delays):

$$
\dot{x}(t) = f(t, x(t), x(\tau_0(t)), \int_{\theta_0(t)}^{t} \sigma(s, x(s))ds, u(t)), \quad t \in I, \quad (1)
$$

$$
x(t) = \varphi_0(t), \quad t \in [\hat{\tau}, t_0], \quad (2)
$$

where $\tau_0(\cdot) \in D$, $\theta_0(\cdot) \in D$, $\varphi_0(\cdot) \in \Phi$ are given functions, the $n$-dimensional function $f(t, x_1, x_2, x_3, u)$ is defined on $I \times O^2 \times \mathbb{R}^n \times U$ and satisfies the following standard conditions: for each fixed $(x_1, x_2, x_3, u) \in O^2 \times \mathbb{R}^n \times U$ the function $f(\cdot, x_1, x_2, x_3, u) : I \rightarrow \mathbb{R}^n$ is measurable; for each compact set $K \subset O$ there exist functions $m_{f,K}(t)$, $L_{f,K}(t) \in L_1(I, [0, \infty))$ such that for almost all $t \in I$

$$
|f(t, x_1, x_2, x_3, u)| \leq m_{f,K}(t) \forall (x_1, x_2, x_3, u) \in K^2 \times \mathbb{R}^n \times U; |f(t, x_1, x_2, x_3, u) - f(t, y_1, y_2, y_3, u)| \leq L_{f,K}(t) \sum_{i=1}^{3} |x_i - y_i| \forall (x_1, x_2, x_3, y_1, y_2, y_3) \in K^4, \forall (x_3, y_3) \in \mathbb{R}^n \times \mathbb{R}^n, \forall u \in U.
$$

Furthermore, the $n$-dimensional function $\sigma(s, x)$ is defined on $I_1 \times O$ and satisfies the standard conditions.

**Definition 1.** Let $\mu_0 = (\tau_0(\cdot), \theta_0(\cdot), \varphi_0(\cdot)) \in D^2 \times \Phi$ and $u(\cdot) \in \Omega$. A function $x(t) = x(t; \mu_0, u) \in O, t \in I_1$, is called a solution of equation (1), with the initial condition (2) or a solution corresponding to the element $\mu_0$ and $u(t)$, and defined on $I_1$, if $x(t)$ satisfies condition (2), is absolutely continuous on $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$. 

[Image]
**Definition 2.** A control \( u(\cdot) \in \Omega \) is called admissible if there exists the corresponding solution \( x(t), t \in I_1 \).

The set of admissible controls is denoted by \( \Omega_0 \).

**Definition 3.** A control \( u(\cdot) \in \Omega_0 \) is called optimal if

\[
J(u_0(\cdot)) = \inf_{u(\cdot) \in \Omega_0} J(u(\cdot)),
\]

where

\[
J(u(\cdot)) = \int_{t_0}^{t_1} f^0(t, x(t), x(\tau_0(t)), \int_{\theta_0(t)}^t \sigma(s, x(s))ds, u(t))dt
\]

and the scalar function \( f^0(t, x_1, x_2, x_3, u) \) is convex, where \( F \). Then for any \( x \in P \) and \( \dot{x} = f(t, x(t), x(\tau_0(t)), \int_{\theta_0(t)}^t \sigma(s, x(s))ds, u(t)) \in U \).

Theorem 1. Let the following conditions hold:

1) \( \Omega_0 \neq \emptyset \);
2) there exists a compact set \( K_0 \subset O \) such that

\[
x(t) = x(t; \mu_0, u(\cdot)) \in K_0, \ t \in [\hat{t}, t_1], \ \forall u(\cdot) \in \Omega_0;
\]

3) for each \( (t, x_1, x_2, \varphi(\cdot)) \in I \times O \times \Phi \) the set

\[
V_F(t, x_1, x_2, \varphi(t)) = \left\{ F(t, x_1, x_2, \int_{\theta(t)}^t \sigma(s, \varphi(s))ds, u) : u \in U \right\}
\]

is convex, where \( F = (f^0, f) \).

Then for any \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that for every \( (\tau_\delta(\cdot), \theta_\delta(\cdot), \varphi(\cdot)) \in D^2 \times \Phi \) and \( P_\delta(t, x_1, x_2, x_3) = (P^0_\delta, P_\delta) \) satisfying the condition

\[
||\tau_\delta - \tau_0||_I + ||\theta_\delta - \theta_0||_I + ||\varphi_\delta - \varphi_0||_I + ||P_\delta||_{K_1} \leq \delta
\]

there exist an optimal control \( u_\delta(t) \in \Omega_0 \) of the perturbed optimization problem

\[
\dot{x}(t) = f(t, x(t), x(\tau_\delta(t)), \int_{\theta_\delta(t)}^t \sigma(s, x(s))ds, u(t)) + P_\delta(t, x(t), x(\tau_\delta(t)), \int_{\theta_\delta(t)}^t \sigma(s, x(s))ds), t \in I,
\]

\[
x(t) = \varphi_\delta(t), \ t \in [\hat{t}, t_0],
\]

\[
J(u(\cdot); \delta) = \int_{t_0}^{t_1} f^0(t, x(t), x(\tau_\delta(t)), \int_{\theta_\delta(t)}^t \sigma(s, x(s))ds, u(t))dt
\]

\[
+ \int_{t_0}^{t_1} p^0_\delta(t, x(t), x(\tau_\delta(t)), \int_{\theta_\delta(t)}^t \sigma(s, x(s))ds) dt \to \min
\]

and

\[
|J(u_0(\cdot)) - J(u_\delta(\cdot); \delta)| \leq \varepsilon.
\]

Here, the functions \( P_\delta \) satisfy the standard conditions and for each \( P_\delta \) there exist \( m_{P_\delta, K_1}(t) \) and \( L_{P_\delta, K_1}(t) \) functions such that

\[
\int_I \left[ m_{P_\delta, K_1}(t) + L_{P_\delta, K_1}(t) \right] dt \leq C,
\]

where \( K_1 \subset O \) is a compact set containing a neighborhood of \( K_0 \) and \( C > 0 \) is a given number;

\[
||P_\delta||_{K_1} = \sup \left\{ \int_{s_1}^{s_2} P_\delta(t, x_1, x_2, x_3) dt : s_1, s_2 \in I, (x_1, x_2) \in K^2_1, x_3 \in R^n \right\}.
\]

**Remark 1.** Let \( U \) be convex set and let \( F(t, x_1, x_2, x_3, u) = A(t, x_1, x_2, x_3) + B(t, x_1, x_2, x_3)u \) then the condition 3) of the Theorem 1 holds.
Theorem 2. Let the conditions 1) and 2) of the Theorem 1 hold. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $(\tau_0(\cdot), \theta_0(\cdot), \varphi_0(\cdot)) \in D^2 \times \Phi$ and $P_0(t, x_1, x_2, x_3, u)$ satisfying the conditions

$$|\tau_0 - \tau_0| + |\theta_0 - \theta_0| + \|\varphi_0 - \varphi_0\|_1 + \|P_0\|_1 \leq \delta$$

and the set $P_{F+P_0}$ is convex, there exists an optimal control $u_0(\cdot) \in \Omega_0$ of the perturbed optimization problem

$$\dot{x}(t) = f(t, x(t), x(\tau_0(t)), \int_{\theta_0(t)}^{t} \sigma(s, x(s))ds, u(t)) + P_0(t, x(t), x(\tau_0(t)), \int_{\theta_0(t)}^{t} \sigma(s, x(s))ds, u(t)),$$

$$x(t) = \varphi_0(t), \ t \in [\tau, t_0],$$

$$J(u(\cdot); \delta) = \int_{t_0}^{t_1} f^0(t, x(t), x(\tau_0(t)), \int_{\theta_0(t)}^{t} \sigma(s, x(s))ds, u(t)) dt$$

$$+ \int_{t_0}^{t_1} P_0(t, x(t), x(\tau_0(t)), \int_{\theta_0(t)}^{t} \sigma(s, x(s))ds, u(t)) dt \to \min$$

and

$$|J(u_0(\cdot)) - J(u_0(\cdot); \delta)| \leq \varepsilon.$$

Here, the function $P_0(t, x_1, x_2, x_3, u)$ satisfy the standard conditions on the set $I \times O^2 \times R^n \times U$ and

$$\int_I \sup \left\{|P_0(t, x_1, x_2, x_3, u)| + \left|\frac{\partial P_0(\cdot)}{\partial x_1}\right| + \left|\frac{\partial P_0(\cdot)}{\partial x_2}\right| + \left|\frac{\partial P_0(\cdot)}{\partial x_3}\right| : (x_1, x_2, x_3, u) \in K^2 \times R^n \times U \right\} dt \leq \text{const};$$

$$\|P_0\|_1 = \int_I \sup \left\{|P_0(t, x_1, x_2, x_3, u)| : (x_1, x_2, x_3, u) \in K^2 \times R^n \times U \right\} dt.$$

Theorems 1 and 2 are proved by the scheme given in [3] and [4].

Keywords: Optimization, sensitivity, differential equation with prehistory.

AMS Subject Classification: 34K27, 34K35, 49J21.

References


If we study electrostatic plasma oscillations, i.e. when the magnetic field is zero, \( B = 0 \), the motion of the electron component of plasma obeys the following plasma hydrodynamics equations:
\[
\begin{align*}
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) &= 0, \\
\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \nabla) \mathbf{v}_e &= -\frac{e}{m} \mathbf{E} - \frac{1}{n_e m} \nabla p,
\end{align*}
\]
(1)
where \( n_e \) is the number density of electrons, \( \mathbf{v}_e \) is the electron velocity, \( m \) is the mass of an electron, \( e > 0 \) is the absolute value of the elementary charge, \( \mathbf{E} \) is the strength of the electric field, and \( p \) is the pressure. We should also consider Maxwell and Poisson equations for the electric field evolution,
\[
\begin{align*}
\frac{\partial \mathbf{E}}{\partial t} &= 4\pi e(n_e \mathbf{v}_e - n_i \mathbf{v}_i), \\
(\nabla \cdot \mathbf{E}) &= -4\pi e(n_e - n_i),
\end{align*}
\]
(2)
where \( n_i \) is the ion number density and \( \mathbf{v}_i \) is the ion velocity.

In the zeroth approximation only electrons participate in a plasma oscillation, with the number density of ions being approximately constant: \( n_i \approx n_0 = \text{const} \). Thus we may present the electric field in the form,
\[
\mathbf{E} = \mathbf{E}_1 e^{-i\omega_e t} + \mathbf{E}_1^* e^{i\omega_e t} + \cdots,
\]
(3)
where \( \omega_e = \sqrt{4\pi e^2 n_0/m} \) is the Langmuir frequency for electrons and \( \mathbf{E}_1 \) is the amplitude of the electric field. It should be noted that, in the following, we shall study axially and spherically symmetric plasma oscillations. In this case one can find a scalar potential \( \phi_1 \) such as \( \mathbf{E}_1 = -\nabla \phi_1 \) in Eq. (3).

In a realistic situation ions will also participate in a plasma oscillation. Thus the ion density becomes \( n_i = n_0 + n(r, t) \), where \( n \) is the perturbation of the ion density. It leads to the appearance of higher harmonics omitted in Eq. (3). The plasma hydrodynamic equations for the description of the ions evolution have the form [4],
\[
\begin{align*}
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_i) &= 0, \\
\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \nabla) \mathbf{v}_i &= \frac{\mathbf{F}_i}{M},
\end{align*}
\]
(4)
where \( M \) is the ion mass, \( \mathbf{F}_i \) is the force acting of ions. The reason why one can omit the ion pressure term in Eq. (4) will be discussed below.

Using the quasineutrality of plasma we can find that [6]
\[
n_0 + n \approx n_e = n_0 \exp\left(\frac{e\varphi_s - U_{pm}}{T_e}\right),
\]
(5)
where $\varphi_s$ is the slowly varying part of the electric potential, $T_e$ is the electron temperature, and $U_{pm} = |E_1|^2/(4\pi n_0)$ is the potential of the ponderomotive force which acts on a charged particle in a rapidly oscillating electric field given in Eq. (3). Supposing that ions are mainly involved in the slow motion of plasma we get that $\mathbf{F}_i = -e\nabla\varphi_s$. Finally, using Eqs. (1)-(5) one arrives to the system of Zakharov equations [6]. More detailed derivation of the nonlinear plasma evolution equations which include electron-ion and electron-electron interactions can be found in Ref. [4].

It should be noted that Eq. (4) is derived under the assumption of ions having point-like charges. However realistic atmospheric plasma contains mainly nitrogen or oxygen ions, which are diatomic. In this case, the simplified ion hydrodynamics Eq. (4) is incomplete since it does not take into account the internal structure of ions.

A diatomic molecule is nonpolar, i.e. it cannot have an intrinsic EDM because of the symmetry reasons. Nevertheless, this kind of molecules can acquire EDM. $p_i = \alpha_{ij} E_j$, in an external electric field. Here $(\alpha_{ij})$ is the polarizability tensor. Hence the additional force, $F_{pol} = (p\nabla)\mathbf{E}$, will act on this particle placed in an external inhomogeneous electric field. Thus, if we study the plasma with diatomic ions, in Eq. (4) one should replace $F_i = -e\nabla\varphi_s \rightarrow -e\nabla\varphi_s + f_{pol}/n_i$, where $f_{pol}$ is the volume density of the ponderomotive force related to the matter polarization.

If an ion is diatomic and possesses an axial symmetry, one can always reduce the polarizability tensor to the diagonal form, $(\alpha_{ij}) = \text{diag}(\alpha_{\parallel}, \alpha_{\perp}, \alpha_{\parallel})$, where $\alpha_{\perp}$ and $\alpha_{\parallel}$ are transversal and longitudinal polarizabilities. The expression for $f_{pol}$ has the form [5],

$$f_{pol} = \frac{1}{8\pi} \left[ \nabla \left( n_i \frac{\partial \varepsilon}{\partial n_i} E^2 \right) - E^2 \nabla \varepsilon \right] = n_i \left[ \langle \alpha \rangle + \frac{4}{45} \frac{(\Delta \alpha E^2)}{T_i} \right] \nabla E^2, \tag{6}$$

where $\varepsilon$ is the permittivity of the ion component of plasma, $T_i$ is the ion temperature, $\langle \alpha \rangle = (2\alpha_{\perp} + \alpha_{\parallel})/3$ is the mean polarizability of an ion, and $\Delta \alpha = \alpha_{\parallel} - \alpha_{\perp}$. It should be noted that the general expression for the ponderomotive force $f_{pol}$ was derived under the assumption of static fields with $(\nabla \times \mathbf{E}) = 0$. However, as we mentioned above, we study electrostatic plasma oscillations with zero magnetic field. Thus Eq. (6) remains valid.

Combining Eqs. (1)-(6) we get the following nonlinear coupled equations for the amplidute of the electric field,

$$i\dot{\mathbf{E}}_1 + \frac{3}{2} \omega_c r_D^2 \nabla (\nabla \cdot \mathbf{E}_1) - \frac{\omega_c}{2n_0} n \mathbf{E}_1 = 0, \tag{7}$$

and for the perturbation of the ion density,

$$\left( \frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) n = \frac{1}{4\pi M} \nabla^2 |\mathbf{E}_1|^2 - \frac{4}{15} \frac{(\Delta \alpha)^2 n_0}{MT_i} \nabla^2 |\mathbf{E}_1|^4, \tag{8}$$

where $r_D = \sqrt{T_e/4\pi e^2 n_0}$ is the Debye length and $c_s = \sqrt{T_e/M}$ is the sound velocity in plasma. To derive Eq. (8) we take into account that $|\mathbf{E}_1|^4 = 6|\mathbf{E}_1|^4$ while averaging over the time interval $\sim 1/\omega_c$.

The first quadratic term in the rhs of Eq. (8) corresponds to the direct interaction of charged ions with the electric field whereas the second quartic term there, $\sim \nabla^2 |\mathbf{E}_1|^4$, is related to the induced EDM interaction. Hence the contribution of this second term to the Langmuir waves dynamics should be typically smaller. However, as shown in Ref. [2], in some cases it is the EDM term which arrests the collapse of Langmuir waves.

It should be noted that in Eq. (8) we neglect the contribution of the ion temperature to the sound velocity. Such a contribution would correspond to a nonzero ion pressure term in Eq. (4). Since we suppose that $T_i \ll T_e$, we can omit the ion pressure. However we keep the ion temperature in the quartic nonlinear term in Eq. (8). In the rhs of Eq. (8) we also neglect term $\sim -n_0(\alpha)\nabla^2 |\mathbf{E}_1|^2/M$, which is small compared to the contribution of the Miller force. Indeed, the ratio of these terms is $\sim n_0(\alpha)$. We shall use the following values of $n_0$ and $\langle \alpha \rangle$: $n_0 \sim 10^{21}\text{ cm}^{-3}$ and $\langle \alpha \rangle \sim 10^{-24}\text{ cm}^{-3}$. For such a parameters, this ratio $\sim 10^{-3}$, that justifies the validity of Eq. (8).
Let us suggest that the density variation in Eq. (8) is slow, i.e. \( \partial^2 n / \partial t^2 \ll c_s^2 \nabla^2 n \). In this subsonic regime Eqs. (7) and (8) can be cast in a single NLSE,

\[
i \dot{E} + \frac{3}{2} \omega_e r_D^2 \nabla (\nabla \cdot E) + \frac{\omega_e}{T_e} \left( \frac{1}{8 \pi n_0} |E|^2 - \frac{2(\Delta \alpha)^2}{15 T_i} |E|^4 \right) E = 0, \tag{9}\]

which has both cubic and quintic nonlinear terms. Note that in Eq. (9) we omit the index “1” in the amplitude of the electric field, i.e. \( E_1 \equiv E \), in order not to encumber the formulas.

We shall examine axially or spherically symmetric plasma oscillations, i.e. \( E = E e_r \), where \( e_r \) is a unit vector in radial direction and \( E \) is a scalar function. Introducing the following dimensionless variables:

\[
\tau = \frac{15}{128 \pi^2} T_i \frac{1}{T_e (n_0 \Delta \alpha)^2} \omega_e t, \quad x = \frac{1}{8 \pi n_0 \Delta \alpha} \sqrt{\frac{5 T_i}{T_e}} r, \quad \psi = 4 \Delta \alpha \sqrt{\frac{\pi n_0}{15 T_i}} E, \tag{10}\]

we can represent Eq. (9) in the form [2],

\[
\frac{\partial \psi}{\partial \tau} + \psi'' + \frac{d - 1}{x} \psi' - \frac{d - 1}{x^2} \psi + \left( |\psi|^2 - |\psi|^4 \right) \psi = 0, \tag{11}\]

which contains no dimensionless parameters. Here \( d = 2, 3 \) is the dimension of space. Eq. (11) with boundary conditions: \( \psi(0, \tau) = \psi(\infty, \tau) = 0 \).

Eq. (11) can be applied to model a rare natural atmospheric plasma phenomenon called a ball lightning (BL) [1]. There are indications that some BL can be rather powerful energy sources. Thus, if one succeeds to implement a BL in a laboratory, this plasma object can be an alternative source [3]. Analytical and numerical analysis of Eq. (11) will be implemented in one of our forthcoming publications.

**Keywords:** Nonlinear Schrödinger equation, plasma oscillations, electric dipole moment, stable atmospheric plasma structure, Langmuir soliton.

**AMS Subject Classification:** 34B15, 34G20, 35C08, 86-04.

**References**


INVERSE SPECTRAL PROBLEM
FOR HILL OPERATOR ON LASSO GRAPH

RAKIB F. EFENDIEV

1Baku Engineering University, Department of Mathematics Teaching, Khirdalan city, H. Aliyev str. 120, Baku, Absheron, AZ0101, Azerbaijan
e-mail: refendiyev@beu.edu.az

ABSTRACT. In this paper, we investigate a generalization of the classical Hill problem with complex potentials to lasso graph. The definition of the Hill operator on lasso graph is given and derived its spectral properties. We solved the inverse problem, proved the uniqueness theorem and provided a constructive procedure for the solution of the inverse problem.

Key words: Lasso graphs, inverse spectral problems, Hill operator, reflection coefficient.

AMS classification: 34B24, 34L05, 34L25, 81U40.

1. INTRODUCTION AND MAIN RESULTS

The main purpose of the present work is to solve the inverse problem for the complex Hill operator on the lasso graph. By lasso graph, we shall understand half-line attached to a loop.

Such problems arise, for example, in theoretical physics when we attempt to investigate the behavior of subatomic particles in which the Feynman diagrams, the graphical representation of mathematical expressions, are used.

To write the mathematical form of the quantum-mechanical amplitude for a process that can be performed by electrons and photons, we first draw a diagram and then use the diagram for that process to occur. If we are interested in the electromagnetic phenomena from macroscopic distances up to regions, several hundred times smaller than the proton, then we use quantum mechanical solutions of these equations that provide us with detailed information.

It is well known that the calculation of probability amplitudes in theoretical particle physics requires using large and complicated integrals. These integrals may be represented graphically as Feynman diagrams. If we have considered different kinds of Feynman diagrams, we will meet with diagrams that have closed loops. It turns out that loop diagrams are rather special and introduce a few deep issues.

Since the quantum graphs are the corresponding generalization of Feynman diagrams that provide for this theory an equally procedure for calculation, our aim here will be to investigate one more solvable model of this type, namely lasso graph.

Let there be given the non-compact graph $G$ where an edge is attached to a loop. The compact part of the graph is a loop with 0 corresponding to the attachment point and whose length we shall, to be definite, take equal to $\pi$, non-compact part is the ray $[0, \infty)$. Consider the following differential equations

$$-y_k''(x_k, \lambda) + q_k(x_k)y_k(x_k, \lambda) = \lambda^2 y_k(x_k, \lambda), k = 1, 2$$

$(1)$
in the space $L_2(G) = L_2(R_+) \oplus L_2(0, \pi)$.

We assume that the potential

$$q(x) = [q_1(x_1), q_2(x_2)],$$

is defined as complex valued, periodic function on the $G$ of the form

$$q_k(x_k) = \sum_{n=1}^{\infty} q_{kn} e^{inx_k}, \quad \sum_{n=1}^{\infty} |q_{kn}| < \infty; \quad k = 1, 2$$

and $\lambda$ is a spectral parameter. We take the potential on the ray to be $q_1(x_1)$ and on the loop $q_2(x_2)$.

Spectral analysis of operator with the periodic potential of the type (2) in $L_2(-\infty, +\infty)$ firstly have been studied by M.G. Gasymov [1].

In particular, M.G. Gasymov in [1] proved the existence of a solution $f(x, \lambda)$ of the equation

$$-y''(x) + q(x)y(x) = \lambda^2 y(x)$$

in $L_2(-\infty, +\infty)$ of the form

$$f(x, \lambda) = e^{i\lambda x} \left( 1 + \sum_{n=1}^{\infty} \frac{1}{n + 2\lambda} \sum_{\alpha = n}^{\infty} V_{n\alpha} e^{i\alpha x} \right).$$

He also discussed the corresponding inverse spectral problem of finding the potential $q(x)$ for given so-called “normalizing” numbers $V_{nn}$ where key role played relation

$$\lim_{\lambda \to \frac{1}{2}} (n - 2\lambda) f(x, -\lambda) = V_{nn} f\left(x, \frac{n}{2}\right).$$

Let us denote by

$$y_1(x_1, \lambda) = [y_{11}(x_1, \lambda), y_{21}(x_1, \lambda)]$$

the solution of the equation (1) on the ray $N_1 = [0, \infty)$, and by the

$$y_2(x_2, \lambda) = [y_{12}(x_2, \lambda), y_{22}(x_2, \lambda)]$$

the solution on the loop that is parametrized to $N_2 = [0, \pi]$.

The functions $y_k, y'_k, k = 1, 2$ are absolutely continuous on $G$ and satisfy the following matching conditions:

1. The continuity condition reads

$$y_1(0, \lambda) = y_2(0, \lambda) = y_2(\pi, \lambda). \quad (3)$$

2. The current conservation is

$$y'_1(0, \lambda) + y'_2(0, \lambda) - y'_2(\pi, \lambda) = 0. \quad (4)$$

The matching conditions (3-4) are called standard conditions and have a definite physical sense, for example, in elastic string networks it express the balance of tension.

Note that the boundary-value problem (1)-(4) admits a natural operator interpretation.

In the Hilbert space

$$H = L_2(G) = L_2(R_+) \oplus L_2(0, \pi)$$

with standard scalar product $(., .)_H$ we define non-self-adjoint operator $A$

$$A \left( \begin{array}{c} y_1(x) \\ y_2(x) \end{array} \right) = \left( \begin{array}{c} -y''_1(x) + q_1(x)y_1(x) \\ -y''_2(x) + q_2(x)y_2(x) \end{array} \right)$$

with domain

$$D(A) = \left\{ \begin{array}{c} y_1(x) \\ y_2(x) \end{array} \right\}: \quad \begin{array}{l} y_1(x) \in W^2_2(0, \infty), y_2(x) \in W^2_2(0, \pi) \\ y_1(0) = y_2(0) = y_2(\pi) \\ y'_1(0) + y'_2(0) - y'_2(\pi) = 0. \end{array} \right\}.$$
Then $A$ is a densely defined closed operator in $H$ and the considered problem (1)-(4) can be interpreted as a study of the operator $A$ on loop like graph introduced as above.

In the present paper we put emphasis on the analytically solution of the problem. Moreover, the potential on graphs with loop (including the potential on loop edge) can be constructed by reflection coefficients and two spectra. In order to solve the inverse problem effective algorithm is given.

**Proposition 1.** For $q(x) = [q_1(x_1), q_2(x_2)]$ defined as (2) there exists a unique solution

$$Y(x, \lambda) = [y_1(x_1, \lambda), y_2(x_2, \lambda)]$$

of the spectral problem (1-4) and a unique reflection coefficients $R_{jj}(\lambda), j = 1, 2$ associated to it.

This means that for $\lambda \neq \pm \frac{n}{2}, n \in N$ there exists a unique function $Y(x, \lambda)$ and unique constant $R_{jj}(\lambda), j = 1, 2$ satisfying

1. $-y''_k(x_k, \lambda) + q_k(x_k) y_k(x_k, \lambda) = \lambda^2 y_k(x_k, \lambda), k = 1, 2$
2. $Y(x, \lambda) = [y_1(x_1, \lambda), y_2(x_2, \lambda)]$ satisfy the boundary conditions (3) and (4);
3. $y_{jk}(x_k, \lambda)$ is a solution on the edge $N_k, k = 1, 2$;

$$y_{jk}(x_k, \lambda) = T_{jk}(\lambda) f^+_k(x_k, \lambda), k \neq j$$

and

$$y_{kk}(x_k, \lambda) = f^+_k(x_k, \lambda) + R_{kk}(\lambda) f^-_k(x_k, \lambda), k = 1, 2.$$

Here $T_{kj}(\lambda)$ are the transmission coefficients , $R_{kk}(\lambda)$ are the reflection coefficients for the equation (1) and

$$f_k(x_k, \lambda) = e^{i\lambda x_k} \left( 1 + \sum_{n=1}^{\infty} \sum_{\alpha=-\infty}^{\infty} \frac{V^{(k)}_{n\alpha}}{n + 2\lambda} e^{i\alpha x_k} \right), k = 1, 2,$$

where the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=-n+1}^{\alpha} \alpha (\alpha - n) \left| V^{(k)}_{n\alpha} \right| : \sum_{n=1}^{\infty} \left| V^{(k)}_{n\alpha} \right|, k = 1, 2,$$

converge and the numbers $V^{(k)}_{n\alpha}$ determines by the following recurrent relations

$$\begin{cases}
\alpha (\alpha - n) V^{(k)}_{n\alpha} + \sum_{s=1}^{\alpha-1} q_{ks-s} V^{(k)}_{n s} = 0, & 1 \leq n < \alpha, \\
\alpha \sum_{n=1}^{\alpha} V^{(k)}_{n\alpha} + q_{k\alpha} = 0, & k = 1, 2.
\end{cases}$$

**References**

THE OPTIMAL PROBLEM RELATED WITH THE CHANGING THE BODY SHAPE

H.J. EFENDIYeva1, L.A. RUSTAMOVA1

1Department of Informatics, Baku State University, Baku, Azerbaijan
e-mail: rasy11@rambler.ru, lam1488@yandex.ru

In this work the problem related with the changing the body shape is considered, where the pairs of the domain is characterized the body shape. These problems are the diffusion processes, expansion problems, elasticity problems, ecological problems, the problem of spreading an oil slick on the sea surface, biological processes etc.

As a rule, these problems are studied by changing the points of the body relative to time. It is often of interest not to change the points of the body, but change its shape. This is due to the determination of the velocity of change of the domain characterizing the body shape.

For the solution of such problems, the change velocity of the shape of the domain and the linear space of convex sets is determined. Changing the domain make it possible to investigate a wide class of practical problems, as optimal control problems.

Let $M$ the totality of convex closed bounded sets in $\mathbb{R}^n$. The function

$$P_D(x) = \sup_{\ell \in D} \langle \ell, x \rangle, \quad x \in D$$

(1)

is called support function of a set $D \in M$, where $P_D(x)$ is continuously convex and positively homogeneous ([2]). The formula (1) compares to each convex closed bounded $D \in M$ convex, continuous positive homogeneous function $P_D(x)$, the converse is also true; such that $P(x) = P_D(x)$ ([2]). The set $D$ coincides with the sub-differentials of the function $P(x)$ at the point $0 \in \mathbb{R}^n$ ([6]).

Let $a = (A_1, A_2), \ b = (B_1, B_2) \ A_i, B_i \in M, \ i = 1, 2. B$ - the unit ball, $B \in S_B$ - the unit sphere.

In [2] it is shown that space $M \times M$ is linear. a scalar product $a \cdot b$ in $M \times M$ is defined as follows

$$a \cdot b = \int_{S_{B}} P(x) \varphi(x) \, ds.$$

Here $P(x) = P_{A_1}(x) - P_{A_2}(x), \ \varphi(x) = P_{B_1}(x) - P_{B_2}(x), \ P_{A_i}(x), P_{B_i}(x)$ is the support function of the sets $A_i$ and $B_i, \ i = 1, 2$ correspondingly.

Suppose that at time $t \in [0; T]$ the domain has a form $D(t)$. By changing the $t$, $D(t)$ is also changes. Then the velocity of change $D(t)$ has the following value

$$\frac{\partial P_{D(t)}}{\partial t} = \lim_{\Delta t \to 0} \frac{P_{D(t+\Delta t)}(x) - P_{D(t)}(x)}{\Delta t}, \quad x \in S_B.$$

If there exist domains $V_1(t), V_2(t) \in M, \ t \in [0; T]$, such that
The functional can be written in the following form:

\[
\frac{\partial P_{D(t)}(x)}{\partial t} = P_{V_1(t)}(x) - P_{V_2(t)}(x),
\]

then the value \( D(t) = (V_1(t), V_2(t)) \in M \times M \), will call the velocity of change of the domain \( D(t) \).

For example \( D(t) = B_t \) is a ball with radius \( t \), with the center at the beginning of coordinates, then \( P_{D(t)}(x) > t \|x\| \) ([2]). Then \( D(t) = (B, 0) \). If \( D(t) \) is a rectangle \( D(t) = \{(x_1, x_2) : 0 < x_1 < 2t, 0 < x_2 < 2t\} \), then \( D(t) = (D(1), 0) \).

Writing \( D(t) = (D_1(t), 0) - (D_2(t), 0) \) and assuming that \( D_1(t), D_2(t) \in M \times M \), we similarly define \( d(t) = D_1(t) - D_2(t) \in M \times M \).

Let the pair of domains \( d(t) = (D_1(t), D_2(t)) \) characterizing the studied object be a solution of the following problem

\[
d(t) = a(t) d(t) + \nu(t), \quad t \in [0; T],
\]

\[
d(0) = d_0,
\]

where \( T > 0 \) the given number of \( d(0) = (D_1(0), D_2(0)) \), the functions \( a(t), \ t \in [0; T] \) and \( d_0 \in M \times M \) are given and \( \nu(t) \in M \times M \). It is assumed that the function \( a(t) \) is continuous with respect to \( t \) on \([0; T]\).

It is required to find \( \nu(t) = (V_1(t), V_2(t)) \in M \times M \) measurable with respect to \( t \) on \([0; T]\).

So that at time \( T \) the \( d(T) \) it is closer to a predetermined element \( y = (#_1, #_2) \in M \).

Mathematically, this problem is reduced to minimizing the functional

\[
\#(\nu) = \|d(T) - y\|_{ML^2}^2 \to \min
\]

under the conditions (1), (2). This functional can be written in the following form

\[
J(\nu) = \int_{S_B} |P_{d(T)}(x) - P_{y(\#)}(x)|^2 ds,
\]

where \( P_{d(T)}(x) = P_{D_1(T)}(x) - P_{D_2(T)}(x), P_{y(\#)}(x) = P_{y_1(\#)}(x) - P_{y_2(\#)}(x) \).

Here we consider the "perturbed" functional in the following form

\[
J(\nu) = \|d(T) - y\|_{ML^2}^2 + \int_{S_B} \|\nu(t)\|_{ML^2}^2 ds \to \min,
\]

where \( a \geq 0 \) the given number.

Suppose that the set of controls has the form

\[
k = \{\nu = (V_1(t), V_2(t)) \in M \times M, \ V_0 \subset V \subset \tilde{V}_0, \ i = 1, 2, \ \forall t \in [0, T]\},
\]

where \( V_0, \tilde{V}_0 \in M \) the given bounded domains. It is known that \( (A, B) = (C, D) \), then \( P_A(x) - P_B(x) = P_C(x) - P_D(x) \) ([2]). Then \( d(t) = (D_1(t), D_2(t)) \in M \times M \), the equation (1) can be written in the equivalent form

\[
\frac{\partial P_{d(t)}(x)}{\partial t} = a(t) P_{d(t)}(x) - P_{\nu(t)}(x), \quad x \in S_B,
\]

or
\[
\frac{\partial P_{D_1}(t)(x)}{\partial t} - \frac{\partial P_{D_2}(t)(x)}{\partial t} = a(t) \left[ P_{D_1}(t)(x) - P_{D_2}(t)(x) \right] + \left[ P_{V_1}(t)(x) - P_{V_2}(t)(x) \right], \quad x \in S_B.
\]

**Lemma.** For any given \( \nu \in k \), the problem (1), (2) is the unique solution of \( d(t) \in M \times M \), \( t \in [0, T] \).

\[
g^* \cdot \nu^*(t) - 2 \| \nu^*(t) \|^2 = \max_{\nu \in k} \left[ g^* \cdot \nu - a \| \nu \|^2 \right], \quad t \in [0, T].
\]

Here \( g^* = -2 \left[ d^* (T) - y \right] \cdot e^{1 \int_0^T \alpha(\tau) d\tau} \) and \( d^* = d^* (t) \) is a solution of the problem (2), (3) by \( \nu = \nu^*(t) \).

**Keywords:** Optimal problem, body shape, diffusion processes, bounded sets.

**AMS Subject Classification:** 49-XX

**References**

FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL KATUGAMPOLA FRACTIONAL INTEGRAL

SEDEF EMIN

Department of Mathematics, Eastern Mediterranean University, Famagusta, T.R. North Cyprus, via Mersin 10, Turkey
e-mail: sedef.emin@emu.edu.tr

1. INTRODUCTION

In recent years, boundary value problems for nonlinear fractional differential equations have been studied by several researchers. In fact, fractional differential equations have played an important role in physics, chemical technology, biology, economics, control theory, signal and image processing, see [1–12, 14–19] and the references cited therein.

Motivated by the papers [6] and [19], the sufficient conditions of existence (uniqueness) of solutions for the following nonlinear fractional differential equation subject to Katugampola fractional integral conditions [13] are investigated

\[ \begin{aligned}
D^\alpha x(t) &= f(t, x(t)), t \in [0, T], \\
x(0) &= 0, x(T) = \beta \mathcal{I}_q x(\xi), 0 < \xi \leq T, \\
x'(T) &= \gamma \mathcal{I}_q x'(\eta), 0 < \eta \leq T,
\end{aligned} \]  

(1)

where \( D^\alpha \) is the Caputo fractional derivative of order \( 2 < \alpha \leq 3 \) in (1), \( \mathcal{I}_q \) is the Katugampola integral of \( q > 0, \rho > 0 \), \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) is a continuous function in (1) and \( \beta, \gamma \in \mathbb{R} \).

In addition, for existence (uniqueness) of solutions of Caputo type fractional differential equation (1), several fixed point theorems are used such as Banach’s fixed point theorem, the Leray-Schauder nonlinear alternative and the Krasnoselskii’s fixed point theorem. Also, some illustrating examples are studied. The following lemma is about solution of nonlinear boundary conditions of Caputo type fractional differential equation.

**Lemma 1.** For any \( y \in C([0, T], \mathbb{R}) \), the following linear fractional boundary value problem

\[ \begin{aligned}
^cD^\alpha x(t) &= y(t), 2 < \alpha \leq 3, \\
x(0) &= 0, x(T) = \beta \mathcal{I}_q x(\xi), 0 < \xi \leq T, \\
x'(T) &= \gamma \mathcal{I}_q x'(\eta), 0 < \eta \leq T
\end{aligned} \]  

(2)

is equivalent to fractional integral equation

\[ x(t) = J^\alpha y(t) + \frac{t}{\Delta} \left( 2\omega_2 (\gamma, \eta) + t\omega_1 (\gamma, \eta) \right) \beta \mathcal{I}_q J^\alpha y(\xi) \]

\[ + \frac{t}{\Delta} \left( -\omega_3 (\beta, \xi) + t\omega_2 (\beta, \xi) \right) \gamma \mathcal{I}_q J^{\alpha-1} y(\eta) \]

\[ - \frac{t}{\Delta} \left( 2\omega_2 (\gamma, \eta) + t\omega_1 (\gamma, \eta) \right) J^\alpha y(T) + \frac{t}{\Delta} \left( \omega_3 (\beta, \xi) - t\omega_2 (\beta, \xi) \right) J^{\alpha-1} y(T), \]

(3)

where

\[ \Delta = 2\omega_2 (\beta, \xi) \omega_2 (\gamma, \eta) + \omega_3 (\beta, \xi) \omega_1 (\gamma, \eta) \neq 0 \]  

(4)
and
\[ \omega_1(\beta, \xi) = \left( \beta \frac{\xi^q}{p^q} \frac{1}{\Gamma(q + 1)} - 1 \right), \quad \omega_2(\gamma, \eta) = \left( T - \gamma \eta \prod_{j=1}^{q} \frac{\eta^j}{1 + \rho j} \right), \]
\[ \omega_3(\beta, \xi) = \left( T^2 - \beta \frac{\xi^2}{2 \prod_{j=1}^{q} \frac{\xi^j}{2 + \rho j}} \right). \]  

Also, the notations
\[ \Omega := \frac{T}{\Gamma((\alpha + 1)(\alpha + 1))} \left( 2 |\omega_2(\gamma, \eta)| + T |\omega_1(\gamma, \eta)| \right) \]
\[ + \frac{T}{\Gamma((\alpha + 1)(\alpha + 1))} \left( 2 |\omega_2(\beta, \xi)| + T |\omega_1(\beta, \xi)| \right) \]
\[ + \frac{T}{\Gamma((\alpha + 1)(\alpha + 1))} \left( |\omega_3(\beta, \xi)| + T |\omega_2(\beta, \xi)| \right) \]
\[ + \frac{T}{\Gamma((\alpha + 1)(\alpha + 1))} \left( |\omega_3(\beta, \xi)| + T |\omega_2(\beta, \xi)| \right) \]
\[ + \frac{T}{\Gamma((\alpha + 1)(\alpha + 1))} \left( |\omega_3(\beta, \xi)| + T |\omega_2(\beta, \xi)| \right) \]
\[ + \frac{T}{\Gamma((\alpha + 1)(\alpha + 1))} \left( |\omega_3(\beta, \xi)| + T |\omega_2(\beta, \xi)| \right) \]
are defined.

In the following theorems, existence (uniqueness) results for the boundary value problem by using Banach’s fixed point theorem, the Leary-Schauder nonlinear alternative, and Krasnoselskii’s fixed point theorem are proved.

**Theorem 1.** (Banach fixed point Theorem) Let \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) be a continuous function. Assume that
\[ (A_1) \quad |f(t, x) - f(t, y)| \leq L \|x - y\| \text{ for all } t \in [0, T], L > 0, x, y \in \mathbb{R} \]
\[ (A_2) \quad L \Omega_1 < 1. \]
Then the boundary value problem (1) has a unique solution on \([0, T] \)

**Theorem 2.** Let \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) be a continuous function. Assume that
\[ (A_3) \text{ there exists a nonnegative function } \phi \in C([0, T], \mathbb{R}) \text{ and a nondecreasing function } \Psi : [0, \infty) \to (0, \infty) \text{ such that} \]
\[ |f(t, u)| \leq \phi(t)\Psi(\|u\|) \text{ for any } (t, u) \in [0, T] \times \mathbb{R}; \]
\[ (A_4) \text{ there exist a constant } M > 0 \text{ such that} \]
\[ \frac{M}{\Psi(M) \|\phi\| \Omega_1} > 1, \]
where \( \Omega_1 \text{ in (7).} \)
Then the problem (1) has at least one solution on \([0, T] \).

**Theorem 3.** Let \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) be a continuous function and condition \((A_2)\) holds. In addition, the function \( f \) satisfies the assumptions:
\[ (A_5) \text{ there exist a nonnegative function } \phi \in C([0, T], \mathbb{R}) \text{ such that} \]
\[ |f(t, u)| \leq \phi(t) \text{ for any } (t, u) \in [0, T] \times \mathbb{R}; \]
(A6) $L\Omega < 1$ where $\Omega$ is defined in (6).

Then the boundary value problem (1) has at least one solution on $[0, T]$.

**Keywords:** Fractional differential equations, Caputo fractional derivative, Katugampola derivative, integral boundary conditions.

**AMS Subject Classification:** 34A08, 34B10.

**References**

1. Introduction

Vulnerability and reliability parameters measure the resistance of the network to disruption of operation after the failure of certain stations or communication links in a communication network. An edge subversion strategy of a graph $G$, say $S$, is a set of edge(s) in $G$ whose adjacent vertices which is incident with the removal edge(s) are removed from $G$. The survival subgraph is denoted by $G - S$. The edge-neighbor-rupture degree of connected graph $G$, $ENR(G)$, is defined to be $ENR(G) = \max \{w(G - S) - |S| - m(G - S), S \subseteq E(G), w(G - S) \geq 1\}$ where $S$ is any edge-cut-strategy of $G$, $w(G - S)$ is the number of the components of $G - S$, and $m(G - S)$ is the maximum order of the components of $G - S$. In this paper we give some results for the edge-neighbor-rupture degree of the graph operations and thorny graph types are examined.

2. Main results

A communication network can be broke down to pieces partially or completely from unexpected factors. This situation can prevent data transmit so there would be a big problem on the system to perform its task. Therefore, the vulnerability and the reliability measure the resistance of the network disturbance of operations after the failure of certain stations. To measure the vulnerability and the reliability we have some parameters which are connectivity [1, 2, 3], integrity [4], scattering number [5], rupture degree [6], neighbor-rupture degree [7] and edge-neighbor-rupture degree [8]. In this section some theorems are given for edge-neighbor-rupture degree on the graph operations. Connected, undirected, simple graphs are examined.
**Theorem.** Let $G$ be a regular graph and $G^*$ is a thorny graph of $G$ (adding a vertex to any vertex of a graph). Then the edge-neighbor-rupture degree of $G$ is,$$\text{ENR}(G^*) = \text{ENR}(G) + 1$$

**Proof.** Since $G$ is regular graph, you can start from any edge to delete. There are two cases.

*Case 1:* If we start to delete an edge that is incident to an added vertex, while $S$ edge-cut-strategy number is not changing, $(G - S)$ numbers of components are increased 1. So the result is increased 1.

*Case 2:* If we start to delete an edge that is not incident to an added vertex, while $S$ edge-cut-strategy number and the number of the components are not changed, maximum order of the components are increased. So the result is increased. The result takes maximum value in case 1. So the proof is completed.

### 3. Numerical results

In this section, an algorithm is proposed in order to calculate the edge-neighbor-rupture degree for any simple finite undirected graph without loops and multiple edges by using the `findENR` function.

**Algorithm** Edge Neighbor Rupture (ENR)

**Output:** ENR value for given any graph $G$

**Begin**

$ENR \leftarrow -\infty$;
for all edge subsets $E_S \subseteq E$ do
if $\text{findENR}(G, E_S) > ENR$ then
$ENR = \text{findENR}(G, E_S)$;
end
end
end.

The function below, `findENR`, returns the ENR value for the edge subset for the graph.

**function** `findENR(G, E_S)`;

**Input:** Graph $G(V, E)$, edge subset $E_S$

**Output:** ENR value for given an $E_S$ edge subset of $G$.

**Begin**

$V_S : \text{vertex set incident with } E_S \text{ edges.}$
for all $u \in V_S$ do
remove $u$ from $G$ i.e. $G - V_S$
end
Componentnumber $\leftarrow \text{find the number of components of } G - V_S$
MaxCompVertexnum $\leftarrow \text{find the vertex number of maximum component of } G - V_S$
findENR $\leftarrow \text{Componentnumber} - \text{number of } (E_S) - \text{MaxCompVertexnum}$
end.

**Keywords:** Edge-neighbor-rupture degree, thorny graphs, vulnerability, reliability.
AMS Subject Classification: 2010, 05C76, 05C85, 68R10.

REFERENCES

SOLVING FREE BOUNDARY PROBLEM FOR AN INITIAL CELL LAYER IN MULTISPECIES BIOFILM FORMATION BY NEWTON-RAPHSON METHOD

MOHAMMAD ASADPOUR FAZLALLAHI¹, KARIM IVAZ¹

¹Faculty of Mathematical science, University of Tabriz, Tabriz, Iran
e-mail: m.asadpour@tabrizu.ac.ir, ivaz@tabrizu.ac.ir

Abstract. The initial attached cell layer in multispecies biofilm growth is considered. The corresponding mathematical model leads to discuss a free boundary problem for a system of nonlinear hyperbolic partial differential equations, where the initial biofilm thickness is equal to zero. No assumptions on initial conditions for biomass concentrations and biofilm thickness are required. The data that the problem needs are the concentration of biomass in the bulk liquid and biomass flux from the bulk liquid. The differential equations are converted into an equivalent system of Volterra integral equations. We use Newton-Raphson method to solve the nonlinear system of Volterra integral equations (SVIEs) of the second kind. This method converts the nonlinear system of integral equations into a linear integral equation at each step.

Keywords: Biofilm, Newton-Raphson method, free boundary problem, nonlinear system of Volterra integral equations.

AMS Subject Classification: 45G15, 92B05.

1. Introduction

Mathematical modelling of biofilm growth was extensively performed during the past decades. Essentially, two different classes of models have been developed: continuum models, e.g. among others [5, 7], and differential-discrete models, e.g. [2, 6]. In principles, methods of statistical mechanics can be used to derive macroscopic equations from the underlying description at the cellular scale [3].

Usually, an initial nonzero thickness in biofilm growth is assumed, and the formation of attached cell layer is neglected, Fig.1(a) and (b). Nevertheless, this biological process can last several days or months, since it depends on many factors such as physical and chemical characteristics of substratum, nutrient concentration, hydrodynamic conditions and concentration of planktonic bacteria in the bulk. Therefore, the formation of attached cell layer is very important in environmental industrial application for wastewater treatment, in particular in the start-up of fixed-growth treatment reactors.

The mathematical model is introduced by Berardino DAcunto, Luigi Frunzo in 2012, where the complete free boundary problem is described [4]. The differential equations are converted into an equivalent system of Volterra integral equations. Subsequently, an existence and uniqueness theorem is proved by the classical fixed point theorem and suitable weighted norms. They show that the solutions are positive and the sum of fraction volumes is equal to 1. In addition, it is proved that the free boundary is an increasing function of time [4]. (see Fig.2)
In this paper, we solve Free boundary problem for an initial cell layer in multispecies biofilm formation by Newton-Raphson method. At first, we introduce linear operator $F$ on system of integral equations, then obtain Frechet derivative. Therefore, one can write iterative formula of Newton-Rafson Method. We show that Contorovich theorem’s conditions satisfies on Newton-Rafson formula.

![Schematic biofilm formation](image1)

**Figure 1.** Schematic biofilm formation. (a) Planktonic cells; (b) Attached cell layer; (c) Cell proliferation; (d) Mature biofilm; (E) Detachment.

![Free boundary problem](image2)

**Figure 2.** Free boundary problem.

### 2. Preliminaries

According to [4], the initial growth process for multispecies biofilms in one space dimension may be described by free boundary problem. By converting differential equations into an equivalent system of Volterra integral equations; one can rewrite as follow:

$$x_i(t_0, t) = \psi_i(t_0) + \int_{t_0}^{t} \Phi_i(X(t_0, \tau), c(t_0, \tau), \tau) d\tau,$$

$$c(t_0, t) = \int_0^{t_0} \sigma(\theta) d\theta + \int_0^{t_0} d\theta \int_0^{\theta} \Phi_{n+1}(X(\tau, \theta), c(\tau, \theta), c_r(\tau, \theta), \theta) d\tau$$

$$+ \int_{t_0}^{t} d\theta \int_0^{t_0} \Phi_{n+1}(X(\tau, \theta), c(\tau, \theta), c_r(\tau, \theta), \theta) d\tau$$

$$c_{t_0}(t_0, t) = \sigma(t_0) + \int_{t_0}^{t} \Phi_{n+2}(X(t_0, \theta), c(t_0, \theta), c_{t_0}(t_0, \theta), \theta) d\theta$$
\[ L(t_0) = \int_0^{t_0} \sigma(\theta)d\theta + \int_0^{t_0} d\theta \int_0^{\theta} G(c(\tau, \theta), \theta, X(\tau, \theta))c_\theta(\tau, \theta)d\tau, \]  

where \( i = 1, \ldots, n \) and \( 0 \leq t_0 \leq t \leq T \). Note that equation (4) is separated from system (1), (2), (3). Thus, this system is solved firstly. Then, the solution is used in equation (4) to find \( L(t) \). Now according with Volterra system (1), (2), (3), we have

\[
\vec{X} = G_0 + \int_{t_0}^{t} \kappa(t, \tau, \vec{X}(\tau))d\tau
\]

where \( 0 < \theta \leq t_0 < t \), and,

\[
\kappa(t, \tau, \vec{X}(\tau)) = \left[ \begin{array}{c}
\Phi_1(\vec{X}(c(t_0, \tau), \tau), c(t_0, \tau), \tau) \\
\Phi_2(\vec{X}(c(t_0, \tau), \tau), c(t_0, \tau), \tau) \\
\vdots \\
\Phi_n(\vec{X}(c(t_0, \tau), \tau), c(t_0, \tau), \tau) \\
\int_0^{t_0} \Phi_{n+1}(\vec{X}(c(\theta, \tau), \tau), c(\theta, \tau), c_\theta(\theta, \tau), \tau)d\theta \\
\Phi_{n+2}(\vec{X}(c(t_0, \tau), \tau), c(t_0, \tau), c_{t_0}(t_0, \tau), \tau)
\end{array} \right]
\]

One can define linear operator \( F \) as follow:

\[
F = \vec{X} - G_0 - \int_{t_0}^{t} \kappa(t, \tau, \vec{X}(\tau))d\tau
\]

and obtain Fr`echet derivative. Then,

\[
F'(\vec{X}(t))v(t) = v(t) + \int_{t_0}^{t} \frac{\partial \kappa}{\partial \vec{X}}(t, \tau, \vec{X}(\tau))v(\tau)d\tau.
\]

By using Newton- Raphson’s method to linearize the problem (5), and applying (7) and (6), we have

\[
\delta_n(t) + \int_{t_0}^{t} \frac{\partial \kappa}{\partial \vec{X}}(t, \tau, \vec{X}_n(\tau))\delta_n(\tau)d\tau = -\vec{X}_n(t) + G_0 + \int_{t_0}^{t} \kappa(t, \tau, \vec{X}_n(\tau))d\tau.
\]

Where, \( \delta_n(t) = X_{n+1}^-(t) - \vec{X}_n(t) \) and \( n = 1, 2, \ldots \).

According to (8), \( \delta_n(t) \) is only unknown function. By choosing suitable initial function \( \vec{X}_0(t) \), system of linear integral equations are solved by commonly numerical methods. Thus, one can obtain \( \vec{X}_1(t) \) and so on.

References

THE PROBLEM OF OPTIMAL SYNTHESIS OF THE 3D-MULTIDIMENSIONAL NONLINEAR MODULAR DYNAMIC SYSTEMS

F.G. FEYZIEV\(^1\), G.H. MAMMADOVA\(^2\), M.R. MEKHTIEVA\(^3\)

\(^1\)Sumgayit State University, Baku, Azerbaijan
\(^2\)Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
\(^3\)Baku State University, Baku, Azerbaijan
e-mail: FeyziyevFG@mail.ru

The statement and solution of the problem of optimal synthesis of the 3D-multidimensional nonlinear modular dynamic systems (3D-MNMSDS), given in the form of a two-digit analogue of the Volterra polynomial are considered. The problem is reduced to the problem of quadratic optimization. A matrix form of this synthesis problem is constructed.

In the work the statement and solution of the problem of optimal synthesis of the 3D-multidimensional nonlinear modular dynamic systems (3D-MNMSDS) with \(n_0\) memory, \(P = P_1 \times P_2\) bounded relation and with a maximum exponent of the degree of nonlinearity \(S\) is considered. This system over the field \(GF(2)\) is described by the following form of two-digit analog of the Volterra polynomial [1, 2]:

\[
y_{\nu}[n, c_1, c_2] = \sum_{i=0}^{S} \sum_{\eta \in \Lambda(i)} h_{i, \nu, \mu, \tilde{\gamma}}[\gamma_1, \gamma_2, \tilde{m}] \times \\
\prod_{\ell \in Q_0(\tilde{\eta})}^{m_{x, \alpha, x, \tilde{\beta}}} u_{[n - \tau(\alpha_\ell, \beta_\ell, \sigma), c_1 + p_1(j_\alpha(\ell)), c_2 + p_2(\mu_\beta(\ell))]},
\]

\(GF(2), \quad \nu = 1, k.\) (1)

Here \(n \in Z_0; \quad c_\alpha \in Z, \quad \alpha = 1, 2; \quad P_\alpha = \{p_\alpha(1),..., p_\alpha(r_\alpha)\}, \quad p_\alpha(1) < ... < p_\alpha(r_\alpha), \quad p_\alpha(\beta) \in Z, \\beta = 1, r_\alpha, \quad \) and besides \(p_\alpha(1), p_\alpha(r_\alpha)\) finite integer numbers \(\alpha = 1, 2; \quad y[n, c_1, c_2] \in GF^k(2)\) and \(u[n, c_1, c_2] \in GF^r(2)\) are output and input sequences of 3D-MNMSDS, where

\[
y[n, c_1, c_2] = (y_1[n, c_1, c_2],..., y_k[n, c_1, c_2]), u[n, c_1, c_2] = (u_1[n, c_1, c_2],..., u_r[n, c_1, c_2]);
\]

\[
\Lambda(i) = \{\tilde{\eta} = (\eta_1, ..., \eta_r) \mid \eta_1 + ... + \eta_r = i, \ \eta_\alpha \in \{0, 1, ..., (n_0 + 1)r_1r_2\}, \alpha = 1, r\};
\]

\[
Q_0(\tilde{\eta}) = \{\ell \mid j \in \{1, ..., r\} \ \eta_\ell \neq 0; \ \eta_\ell \text{ is the component of vector } \tilde{\eta}\};
\]

\[
F(\tilde{\eta}) = \prod_{\ell \in Q_0(\tilde{\eta})}^{m_{x, \alpha, x, \tilde{\beta}}} F_\ell(\eta_\ell), \quad L_1 = \prod_{\ell \in Q_0(\tilde{\eta})}^{} L_{\ell, 1}(\gamma_1(\ell)), \quad L_2 = \prod_{\ell \in Q_0(\tilde{\eta})}^{} L_{\ell, 2}(\gamma_2(\ell));
\]

\(\Gamma(\gamma_1, \gamma_2, \tilde{m}) = \prod_{\ell \in Q_0(\tilde{\eta})}^{\Gamma_{\ell}(\gamma_1(\ell), \gamma_2(\ell), \tilde{m}_\ell)};\)

\[
Q_\ell(\eta_\ell, \gamma_1(\ell), \gamma_2(\ell), \tilde{m}_\ell) = \{(\alpha, \beta)|m_{x, \alpha, \beta}\text{is the component of }\tilde{m}_\ell \text{ and}}
\]

164
The sets $F_\ell(\eta_\ell), L_{\ell,1}(\gamma_1(\ell)), L_{\ell,2}(\gamma_2(\ell)), \ell \in Q_0(\bar{\eta})$, are the following:

$$F_\ell(\eta_\ell) = \{(\gamma_1(\ell), \gamma_2(\ell), \bar{m}_\ell) \mid \bar{m}_\ell = (m_{\ell,1}, \ldots, m_{\ell,\gamma_1(\ell), \gamma_2(\ell)}) ; \sum_{\alpha=1}^{\gamma_1(\ell)} \sum_{\beta=1}^{\gamma_2(\ell)} m_{\ell,\alpha,\beta} = \eta_\ell \} ; \quad m_{\ell,\alpha,\beta} \in \{0, \ldots, n_0 + 1\}, \alpha = \Gamma_1(\ell), \beta = \Gamma_2(\ell);$$

$$(\forall \alpha \in \{1, \ldots, \gamma_1(\ell)\}) (\exists \beta \in \{1, \ldots, \gamma_2(\ell)\}) (m_{\ell,\alpha,\beta} \neq 0)$$

and

$$(\forall \beta \in \{1, \ldots, \gamma_2(\ell)\}) (\exists \alpha \in \{1, \ldots, \gamma_1(\ell)\}) (m_{\ell,\alpha,\beta} \neq 0); \quad \gamma_\sigma(\ell) \in \{1, \ldots, r_\sigma\}, \quad \sigma = \Gamma(2),$$

$L_{\ell,1}(\gamma_1(\ell)) = \{ (\bar{m}_\ell) = (j_1(\ell), \ldots, j_{\gamma_1(\ell)}(\ell)) \mid 1 \leq j_1(\ell) < \ldots < j_{\gamma_1(\ell)}(\ell) \leq \tau_1 \}$,

$L_{\ell,2}(\gamma_2(\ell)) = \{ (\bar{m}_\ell) = (\mu_1(\ell), \ldots, \mu_{\gamma_2(\ell)}(\ell)) \mid 1 \leq \mu_1(\ell) < \ldots < \mu_{\gamma_2(\ell)}(\ell) \leq \tau_2 \}$.

When $\bar{m}_\alpha, \bar{m}_\beta \in \Gamma_{\ell,\alpha,\beta}(m_{\ell,\alpha,\beta})$, $(\alpha, \beta) \in Q_\ell(\eta_\ell, \gamma_1(\ell), \gamma_2(\ell), \bar{m}_\ell)$, $\alpha = \Gamma_1(\ell), \beta = \Gamma_2(\ell)$, the set of all block vectors (sets) $\bar{m}_\ell$ is denoted by $\Gamma_\ell(\gamma_1(\ell), \gamma_2(\ell), \bar{m}_\ell)$, where

$$\Gamma_{\ell,\alpha,\beta}(m_{\ell,\alpha,\beta}) = \{ \tau_{\ell,\alpha,\beta} = (\tau_\ell(\alpha, \beta, 1), \ldots, \tau_\ell(\alpha, \beta, m_{\ell,\alpha,\beta})) \mid 0 \leq \tau_\ell(\alpha, \beta, 1) < \ldots < \tau_\ell(\alpha, \beta, m_{\ell,\alpha,\beta}) \leq n_0 \}.$$
\( i \in \{0, ..., S\} \) which in the event that the sequence (4) arrives at its input, an actual sequence is obtained at the output that coincides with the desired output sequence

\[
\{y_0^n[n, c_1, c_2] : n \in [0, N], c_1 \in [0, C_1], c_2 \in [0, C_2]\}, \nu = 1, k.
\]

(5)

Lets

\[
J = \sum_{n=0}^{N} \sum_{c_1=0}^{C_1} \sum_{c_2=0}^{C_2} \sum_{\nu=1}^{k} (y_0^n[n, c_1, c_2] - y_0^n[n, c_1, c_2])^2.
\]

(6)

The functional (6) indicates the distance between the real output sequence

\[
\{y_0^n[n, c_1, c_2] : n \in [0, N], c_1 \in [0, C_1], c_2 \in [0, C_2]\}, \nu = 1, k,
\]

and the known desired output sequence (5) of 3D-MNMDS (3). Using the functional (6), the binary-3D-MNMDS synthesis problem (3) can be defined as the following quadratic optimization problem:

It is necessary to find the values of impulse characteristics \( h_{i, j, \tilde{\eta}}(\gamma_1, \gamma_2, \tilde{m}) [\tilde{j}, \tilde{\mu}, \tilde{n}] \), \( \tilde{n} \in \Gamma(\gamma_1, \gamma_2, \tilde{m}) \), \( \tilde{j}, \tilde{\mu} \in L_1 \times L_2 \), \( (\gamma_1, \gamma_2, \tilde{m}) \in F(\tilde{\eta}), \tilde{\eta} \in \Lambda, i \in \{0, ..., S\} \), of 3D-MNMDS (3) under which the functional (6) is minimized.

In the work a block vector \( H \) is successively constructed from \( h_{i, j, \tilde{\eta}}(\gamma_1, \gamma_2, \tilde{m}) [\tilde{j}, \tilde{\mu}, \tilde{n}] \), when \( \tilde{n} \in \Gamma(\gamma_1, \gamma_2, \tilde{m}) \), \( \tilde{j}, \tilde{\mu} \in L_1 \times L_2 \), \( (\gamma_1, \gamma_2, \tilde{m}) \in F(\tilde{\eta}), \tilde{\eta} \in \Lambda, i \in \{0, ..., S\} \) and a block matrix \( U \) is successively constructed when \( n \in [0, N] \equiv \{0, 1, ..., N\}, \ c_1 \in [0, C_1] \equiv \{0, 1, ..., C_1\}, \ c_2 \in [0, C_2] \equiv \{0, 1, ..., C_2\}, \ \tilde{n} \in \Gamma(\gamma_1, \gamma_2, \tilde{m}) \), \( \tilde{j}, \tilde{\mu} \in L_1 \times L_2 \), \( (\gamma_1, \gamma_2, \tilde{m}) \in F(\tilde{\eta}), \tilde{\eta} \in \Lambda, i \in \{0, ..., S\} \)

and on the basis of which 3D-MNMDS (3) is represented in the following matrix form:

\[
Y = U \cdot H, \quad GF(2).
\]

(7)

Lets

\[
Y^0 = (y_0^0[0, 0, 0], ..., y_0^0[0, 0, C_2], ..., y_0^0[0, N, C_1, C_2], y_0^0[N, C_1, C_2])^T,
\]

\[
Y = (y_1[0, 0, 0], ..., y_1[0, 0, C_2], ..., y_1[0, N, C_1, C_2], y_1[N, C_1, C_2])^T.
\]

Then the functional (6) is represented in the form:

\[
J = (Y - Y^0)^T \cdot (Y - Y^0) \rightarrow \min.
\]

(8)

The paper considers the development of a method and an algorithm for solving the problem (7), (8).

**Keywords:** Modular dynamic systems, 3D- multidimensional nonlinear modular dynamic systems, Volterra polynomial, the problem of optimal synthesis, the problem of quadratic optimization.

**AMS Subject Classification:** 37N35, 11S31.

**References**


PARAMETRIC IDENTIFICATIONAL DETERMINATION OF THE FUNCTIONS OF RELATIVE PHASE PERMEABILITIES OF GAS-CONDENSATE DEPOSITS IN WATER DISPLACEMENT MODE

KH.A. FEYZULLAYEV¹, M.S. KHALILOV ², E.A. KULIEV¹

¹Oil Gas Scientific Research Project Institute, SOCAR, Baku, Azerbaijan
²Department of Informatics, Baku State University, Baku, Azerbaijan
e-mail: feyzullayevxasay@gmail.ru, khalilov_mubariz@mail.ru

Abstract. The problem of determining the function of relative phase permeability, included into the hydrodynamic model of three-phase filtration of the fluids and the algorithm of its solution is elaborated. As the main criterion was chosen the minimization of the functional defined in different points in time on the difference between the measured and calculated values of pressure in the well.

Keywords: Pressure, porosity, relative permeability, condensate and water mixture, conjugate problem, gradient method.

AMS Subject Classification: 93B30.

It is expected that the gas-condensate field is operated by water displacement mode. In this case, the radial axis-symmetric flat parallel flow of gas-condensate in the Central well is described by the following system of nonlinear differential equations [1]:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left\{ rA \frac{\partial p}{\partial r} \right\} - \frac{\partial B}{\partial t} = 0, \quad (1)
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left\{ (A + C) \frac{\partial p}{\partial r} \right\} - \frac{\partial D}{\partial t} = 0, \quad (2)
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left\{ rE \frac{\partial p}{\partial r} \right\} - \frac{\partial G}{\partial t} = 0, \quad (3)
\]

\[
p(r, t)|_{t=0} = p_0, \quad s_k(r, t)|_{t=0} = 0, \quad s_{cb}(r, t)|_{t=0} = s_{cb}, \quad (4)
\]

\[
2\pi rh(A + C) \frac{\partial p}{\partial r} \bigg|_{r = r_A} = Q_g(t), \quad (5)
\]

\[
p(R_k, t) = p_k(t), \quad s_k(R_k, t) = 0, \quad s_w(R_k, t) = 1. \quad (6)
\]

Here

\[
A = \frac{k_f s_g p \beta [1 - c(p)\eta(p)]}{\mu_g(p)z(p)p_{at}} + \frac{k_f s_g s_w S_k(p)}{\mu_k(p)a_k(p)};
\]

\[
B = \frac{mS_k(p)}{a_k(p)} s_k + (1 - s_k - s_w) m \frac{p \beta [1 - A(p)\eta(p)]}{z(p)p_{at}};
\]
\[ C = \frac{k_f(s_g)pc(p)r}{\mu_g(p)z(p)p_{at}} + \frac{k_f(s_g, s_w)}{\mu_c(p)a_c(p)} \]

\[ D = \frac{m_s}{a_k(p)} + (1 - s_c - s_w) \frac{mp/3c(p)}{z(p)p_{at}} \]

\[ E = \frac{k_f(w(s_w))}{\mu_w(p)a_w(p)} \]

\[ G = \frac{m_s}{a_k(p)} + B = \frac{mp}{z(p)} \]

\[ f_g \] is a relative phase permeability for the gas; \( f_c \) is a relative phase permeability for condensate; \( f_w \) is a relative phase permeability for water; \( \beta \) is the coefficient of temperature; \( c(p) \) is the content of condensation in the gas phase; \( \tau(p) \) is the ratio of unit weights of condensation in liquid and gas phase under normal conditions; \( z(p) \) is the coefficient of supercompressibility for gas phase; \( p_{at} \) is atmospheric pressure; \( S_k(p) \) is the amount of gas dissolved in a liquid; \( \mu_c(p) \) is the viscosity of condensate phase; \( \mu_w(p) \) is the viscosity of water; \( \mu_g(p) \) is the viscosity of the gas; \( a_c(p) \) is the volumetric coefficient of condensate phase; \( a_w(p) \) is the volumetric coefficient of water; \( D_1 \) = \( \{r_c < r < R_k, \quad t \in (0, T)\} \); \( r_c \) and \( R_k \) are the well radius and the radius of drainage wells, respectively.

To identify the functions of the OFP included in system (1)-(6)

\[ f_g(s_g) = \alpha_1 s_g^{\alpha_2}, \quad f_k(s_g, s_w) = \alpha_3 ((1 - s_c) - (s_g + s_w))^{\alpha_4}, \quad f_w(s_w) = \alpha_5 s_w^{\alpha_6}. \]  

It is needed to determine such values of unknown parameters comprising their expressions in which the solution of problem (1)-(6) minimizes a function

\[ J(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = \int_0^T [p(r_c, t) - p_c(t)]^2 dt + \varepsilon \left( \sum_{i=1}^6 \alpha_i^2 \right). \]  

Here \( p(r_c, t) \) are downhole pressure based on the solution of forward tasks; \( p_c(t) \) are actual downhole pressures measured in fisheries; \( \varepsilon \) is a regularization parameter.

With the introduction of unknown functions \( \Psi_1(r, t), \Psi_2(r, t), \Psi_3(r, t) \), the considered problem of minimization (1)-(8) is stated as a variational problem of finding the minimum of the following functional

\[ J(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = \int_0^T [p(r_A, t) - p_A(t)]^2 dt + \int_{D_1} \Psi_1(r, t) \left[ \frac{1}{r} \frac{\partial}{\partial r} \left\{ rA \frac{4p}{4r} \right\} - \frac{4B}{4t} \right] drdt + \]

\[ + \int_{D_1} \Psi_2(r, t) \left[ \frac{1}{r} \frac{\partial}{\partial r} \left\{ rC \frac{4p}{4r} \right\} - \frac{4D}{4t} \right] drdt \]

\[ + \int_{D_1} \Psi_3(r, t) \left[ \frac{1}{r} \frac{\partial}{\partial r} \left\{ rE \frac{4p}{4r} \right\} - \frac{4G}{4t} \right] drdt + \varepsilon \left( \sum_{i=1}^6 \alpha_i^2 \right) \]

for unknown parameters \( \alpha_i \) (i = 1, 6).

Finding the minimum of functional (9) (solution of the identification problem) is done using gradient method [5]. For finding functions \( \Psi_1(r, t) \), \( \Psi_2(r, t) \) and \( \Psi_3(r, t) \) the conjugate problem relevant to the direct problem (1)-(6) is solved. Thus, defining functions \( \Psi_1(r, t) \), \( \Psi_2(r, t) \) and \( \Psi_3(r, t) \) the increment of functional \( J(\alpha) \) is defined by the following expression:

\[ \Delta J(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = \sum_{i=1}^4 \Delta \alpha_i \left\{ \int_{D_1} \Psi_1 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( rA_{\alpha_i} \frac{\partial p}{\partial r} \right) \right] + \Psi_2 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( rC_{\alpha_i} \frac{\partial p}{\partial r} \right) \right] \right\} + \]
\[ + \int_0^T \left( (\Psi_1 A + \Psi_2 C + \Psi_3 E)(A + C)^{-1}(A_{\alpha_i} + C_{\alpha_i}) \frac{\partial p}{\partial r} \right) \bigg|_{r = r_A} \, drdt + 2\varepsilon \alpha_i + \varepsilon \Delta \alpha_i \right) + \\
+ \sum_{i=5}^{6} \Delta \alpha_i \left\{ \int_{D_i} \Psi_3 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r E_{\alpha_i} \frac{\partial p}{\partial r} \right) \right] \, drdt + 2\varepsilon \alpha_i + \varepsilon \Delta \alpha_i \right\}. \] 

Hence, for the gradient of the function \( J \) we have:

\[
\frac{\partial J}{\partial \alpha_i} = \int_{D_1} \left\{ \Psi_1 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r A_{\alpha_i} \frac{\partial p}{\partial r} \right) \right] + \Psi_2 \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r C_{\alpha_i} \frac{\partial p}{\partial r} \right) \right] \right\} \, drdt + \\
+ \int_0^T \left( (\Psi_1 A + \Psi_2 C + \Psi_3 E)(A + C)^{-1}(A_{\alpha_i} + C_{\alpha_i}) \frac{\partial p}{\partial r} \right) \bigg|_{r = r_A} \, drdt + 2\varepsilon \alpha_i, \quad i = 1, 4, \quad (10)
\]

To identify the desired parameters \( \alpha_i (i = 1, 6) \), included in the theoretical expression of the relative phase permittivities, using the received gradient expressions we have the following iterative formulas

\[
\alpha_i^{k+1} = \alpha_i^k - \lambda_{ik} \frac{\partial J(\alpha_i^k)}{\partial \alpha_i}, \quad i = 1, 6. \quad (12)
\]

Here the derivatives \( \frac{\partial J}{\partial \alpha_i} \) are calculated using expression (11).

Thus, an algorithm for solving the inverse problem of the function of relative permeability is as follows:

- the first approximations of the coefficients \( \alpha_i (i = 1, 6) \) are given, direct problem (1)-(6) is solved by finite difference method for a period of time \((0, T)\) of the known history of field development, and there are identified the estimated reservoir pressure distributions and condensate saturation in the deposits in different time layers.

- optimized parameters in iterations are defined by the formula (12).

- iteration procedure of parameter refinement continues until the difference in two adjacent functional iterations will become less than the specified error of calculations.

**Conclusion**

A method is developed for the parametric identification of hydrodynamic model of three-phase filtration for determining the functions of relative phase permittivities of gas condensate and water system on the basis of change of the actual data of indicators of the reservoir development.

**References**


THE PROBLEM OF IDENTIFICATION OF THE KINETIC COEFFICIENT AND THE RIGHT PART IN CONVECTION-REACTION EQUATION

KH.M. GAMZAEV

1Azerbaijan State Oil and Industry University
e-mail: xan.h@rambler.ru

Abstract. The combined inverse problem of restoration of dependences of the reaction kinetic coefficient and the right part of the one-dimensional convection-reaction equation from time is considered. For numerical solution of the obtained problem, a non-iterative computational algorithm is proposed. It is based on reducing of the problem to two direct boundary-value problems and linear equations with respect to the unknown functions.

Keywords: Convection-reaction equation, combined inverse problem, nonlocal integral condition, differential-difference problem, explicit-implicit schemes.

AMS Subject Classification: 65M32.

Let us consider a one-dimensional inhomogeneous convection-reaction equation

\[ \frac{\partial u(x,t)}{\partial t} + \nu(t) \frac{\partial u(x,t)}{\partial x} + k(t)u(x,t) = f(t)q(x,t), \quad 0 < x < l, \quad 0 < t \leq T, \]

where \( \nu(t) > 0 \), with the following initial and boundary conditions

\[ u(x,0) = \varphi(x), \quad 0 \leq x \leq l, \]

\[ u(0,t) = p(t), \quad 0 \leq t \leq T. \]

It is known that the direct problem for equation (1) consists in determining the function \( u(x,t) \) from equation (1) with given coefficients \( \nu(t) \), \( k(t) \), the right part \( f(t)q(x,t) \) and the additional conditions (2), (3). Suppose that in addition to the function \( u(x,t) \) the functions \( k(t) \) and \( f(t) \) are also unknown. It is required to restore these functions under the following additional conditions

\[ u(l,t) = c(t), \quad 0 \leq t \leq T, \]

\[ \int_0^l u(x,t)dx = w(t), \quad 0 \leq t \leq T, \]

where \( c(t) \) and \( w(t) \) given functions. It is guessed that in this case the matching conditions are satisfied.
\[ c(0) = \varphi(0), \quad p(0) = \varphi(0), \quad \int_0^l \varphi(x)dx = w(0). \]

Thus, the identification problem is to determine the functions \( u(x,t) \), \( k(t) \), \( f(t) \) satisfying equation (1) and conditions (2)–(5). In view of the fact that in problem (1)–(5) it is required to restore simultaneously the reaction kinetic coefficient \( k(t) \) and the right part \( f(t) \) in equation (1), the problem posed is considered to be a combined inverse problem [1,2]. However, in this problem the additional condition (5) is not a local condition for equation (1).

We reduce the problem (1)–(5) to the problem with local conditions. We differentiate relation (5) with respect to the variable \( t \)

\[
\int_0^l \frac{\partial u(x,t)}{\partial t} dx = \frac{dw(t)}{dt}.
\]

Substituting the expressions \( \frac{\partial u(x,t)}{\partial t} \) from equation (1) into this relation and integrating by parts, taking into account (3)–(5), we obtain

\[-v(t)c(t) + v(t)p(t) - k(t)w(t) + f(t)g(t) = \frac{dw(t)}{dt},\]

where \( g(t) = \int_0^l q(x,t)dx. \)

Solving the last equation we will have

\[ f(t) = \frac{\frac{dw(t)}{dt} + v(t)(c(t) - p(t)) + k(t)w(t)}{g(t)}. \]

Then equation (1), taking into account (6), can be written in the form:

\[
\frac{\partial u(x,t)}{\partial t} + v(t)\frac{\partial u(x,t)}{\partial x} + k(t)u(x,t) = k(t)b(x,t) + d(x,t), \quad 0 < x < l, \quad 0 < t \leq T,
\]

where \( b(x,t) = w(t)q(x,t)/g(t), \quad d(x,t) = \left( \frac{dw(t)}{dt} + v(t)(c(t) - p(t)) \right) q(x,t)/g(t). \)

The derivative \( \frac{\partial u(x,t)}{\partial t} \) in equation (1) is discretized by the “backward” difference

\[
\frac{\partial u(x,t)}{\partial t} \bigg|_{t=t_j} \approx \frac{u(x,t_j) - u(x,t_{j-1})}{\Delta t}.
\]

Denoting \( u^j(x) = u(x,t_j), \quad k^j \approx k(t_j) \) we write the problem (7), (2)–(4) in the following form

\[
\frac{u^j(x) - u^{j-1}(x)}{\Delta t} + v^j \frac{du^j(x)}{dx} + k^j u^{j-1}(x) = k^j b^j(x) + d^j(x), \quad 0 < x < l,
\]

\[ u^j(0) = p^j, \]

\[ u^j(l) = c^j, \]

\[ j = 1, 2, \ldots, m, \]

\[ u^0(x) = \varphi(x), \]

where \( p^j = p(t_j), \quad v^j = v(t_j), \quad b^j(x) = b(x,t_j), \quad c^j = c(t_j), \quad d^j(x) = d(x,t_j). \)

As suggested in [3,4], suppose that the solution of the obtained differential-difference problem (8)–(11) on each time layer \( j = 1, 2, \ldots, m \) can be represented in the form
\[ u^j(x) = y^j(x) + k^j z^j(x), \quad (12) \]

where \( y^j(x), \ z^j(x) \) – unknown functions. Substituting the relation (12) into (8), (9), we have

\[
\left\{ \begin{array}{l}
\frac{y^j(x) - w^{j-1}(x)}{\Delta t} + v^j \frac{dy^j(x)}{dx} - d^j(x) + k^j \left[ \frac{z^j(x)}{\Delta t} + v^j \frac{dz^j(x)}{dx} + w^{j-1}(x) - b^j(x) \right] = 0. \\
y^j(0) + k^j z^j(0) = p^j,
\end{array} \right.
\]

From the last relations one can obtain the following direct boundary-value problems with respect to the functions \( y^j(x), \ z^j(x) \)

\[
\frac{y^j(x) - w^{j-1}(x)}{\Delta t} + v^j \frac{dy^j(x)}{dx} - d^j(x) = 0, \quad 0 < x < l, \quad (13)
\]
\[
y^j(0) = p^j, \quad (14)
\]
\[
\frac{z^j(x)}{\Delta t} + v^j \frac{dz^j(x)}{dx} + w^{j-1}(x) - b^j(x) = 0, \quad 0 < x < l, \quad (15)
\]
\[
z^j(0) = 0, \quad . \quad (16)
\]

And the substitution of (12) into (10) yields

\[
y^j(l) + k^j z^j(l) = c^j. \quad (17)
\]

From the relations obtained, the following computational algorithm can be constructed to determine \( u^j(x), \ k^j \), \( j = 1, 2, \ldots, m \):

for a fixed value of the time layer \( j \), the solutions of the direct problems (13), (14) and (15), (16) are determined;

from the relation (17) the approximate value of the required function \( k(t) \) is determined for \( t = t_j \):

\[
k^j = (c^j - y^j(l))/z^j(l);
\]

by formula (6), the approximate value of the unknown function \( f(t) \) is determined for \( t = t_j \);

by formula (12), the approximation of the desired function \( u(x, t) \) is determined for \( t = t_j \);

When you move to the next time layer, the described calculation procedure is repeated again.

For the numerical solution of problems (13), (14) and (15), (16), the finite difference method can be used.

\[ \text{References} \]


OPTIMAL CONTROL OF THE QUASI-LINEAR NEUTRAL DIFFERENTIAL EQUATION

NIKA GORGODZE

1Akaki Tsereteli Kutaisi State University, Department of Mathematics, Kutaisi, Georgia
e-mail: nikagorgodze@yahoo.com

1. Statement of the Problem and Necessary Conditions of Optimality

Neutral differential equation is a mathematical model of a dynamic system, the behavior of which in the given moment of time depends on the system state and velocity in the previous moments (in the past). Neutral differential equations have some special features. In general, Cauchy’s problem solution for the nonlinear neutral equation isn’t continuous toward the right-hand side disturbance of the equation [2, 3], and the analogy of Pontryagin’s maximum principle is not true for the nonlinear neutral optimal problem [1].

Let \( \tau_{2i} > \tau_{1i} > 0, i = \overline{1, s} \) be given numbers and let \( I = [t_0, t_1] \) be a given interval; suppose that \( O \subset \mathbb{R}^n \) is a open set and \( U \subset \mathbb{R}^r \) is a bounded set, the \( n \)-dimensional function \( \varphi(t) \in O, t \in [\tau, t_0] \), where \( \tau = t_0 - \max\{\tau_{21}, ..., \tau_{2s}\} \) is continuously differentiable, by \( \Omega \) we denote the set of measurable functions \( u(t) \in U, t \in I \). Furthermore, let \( A_i(t), t \in I, i = \overline{1, k} \) be continuous matrix functions with the dimension \( n \times n \) and let \( n \)-dimensional functions \( f(t, x, y_1, ..., y_s, u) \) be continuous on \( I \times O^{1+s} \times U \) and continuously differentiable with respect to \( x, y_1, ..., y_s \); suppose that the scalar functions \( q^i(\tau_{1i}, ..., \tau_{si}, x), i = \overline{0, t}, (\tau_{1i}, ..., \tau_{si}, x) \in [\tau_{11}, \tau_{21}] \times ... \times [\tau_{1s}, \tau_{2s}] \times O \) are continuously differentiable.

To each element \( w = (\tau_{1i}, ..., \tau_{si}, u(t)) \in W = [\tau_{11}, \tau_{21}] \times ... \times [\tau_{1s}, \tau_{2s}] \times \Omega \) we assign the quasi-linear neutral differential equation

\[
\dot{x}(t) = \sum_{i=1}^{k} A_i(t)\dot{x}(t - \sigma_i) + f(t, x(t), x(t - \tau_1), ..., x(t - \tau_s), u(t))
\]

with the condition

\[
x(t) = \varphi(t), t \in [\tau, t_0],
\]

where \( \sigma_i > 0, i = \overline{1, k} \) are given number.

**Definition 1.** Let \( w \in W \). A function \( x(t) = x(t; w) \in O, t \in [\tau, t_1], \) is called a solution of equation (1) \textit{with the initial condition (2)} or a solution corresponding to the element \( w \) if it satisfies condition (2) and is absolutely continuous on the interval \([t_0, t_1]\) and satisfies equation (1) almost everywhere on \([t_0, t_1]\).

**Definition 2.** An element \( w \in W \) is said be admissible if the corresponding solution \( x(t) = x(t; w) \) satisfies the conditions

\[
q^i(\tau_{1i}, ..., \tau_{si}, x(t_1)) = 0, i = \overline{1, t}.
\]

Denote by \( W_0 \) the set of admissible elements.

**Definition 3.** An element \( w_0 = (\tau_{10}, ..., \tau_{s0}, u_0(t)) \in W_0 \) is said to be optimal if for arbitrary element \( w \in W_0 \) the following inequality holds

\[
q^0(\tau_{10}, ..., \tau_{s0}, x(t_1; w_0)) \leq q^0(\tau_{1i}, ..., \tau_{si}, x(t_1; w)).
\]

The optimal control problem consists in finding the optimal element \( w_0 \).
**Theorem 1.** Let \( w_0 \) be an optimal element and let \( x_0(t) \) be corresponding optimal trajectory, suppose that \( \tau_0 \in (\tau_{1i}, \tau_{2i}) \). Then there exist a vector \( \pi = (\pi_0, \ldots, \pi_i) \neq 0, \pi_0 \leq 0 \) and a solution \((\chi(t), \psi(t))\) of the system

\[
\begin{cases}
\dot{\chi}(t) = -\psi(t)f_x[t] \\
- \sum_{i=1}^k \psi(t + \tau_0)f_y[t + \tau_0], \\
\psi(t) = \chi(t) + \sum_{i=1}^k \psi(t + \sigma_i)A_i(t + \sigma_i), t \in [t_0, t_1], \\
\chi(t) = \psi(t) = 0, t > t_1,
\end{cases}
\]

where \( f_x[t] = f_x(t, x_0(t), x_0(t - \tau_{10}), \ldots, x_0(t - \tau_{s0}), u_0(t)) \), such that the conditions listed below hold:

1) the condition for \( \chi(t) \) and \( \psi(t) \)

\[
\chi(t_1) = \psi(t_1) = \pi Q_{0x},
\]

where

\[
Q = (q^0, \ldots, q^I)^T, Q_{0x} = Q_x(\tau_{10}, \ldots, \tau_{s0}, x_0(t_1)):
\]

2) the condition for the optimal delay \( \tau_0 \)

\[
\pi Q_{0r_i} = \int_{t_0}^{t_1} \psi(t)f_y[t]x_0(t - \tau_0)dt, i = 1, \ldots, s;
\]

3) the integral condition for the optimal control function \( u_0(t) \)

\[
\int_{t_0}^{t_1} \psi(t)f(t, x_0(t), x_0(t - \tau_{10}), \ldots, x_0(t - \tau_{s0}), u_0(t))dt = \max_{u(t) \in U} \int_{t_0}^{t_1} \psi(t)f(t, x_0(t), x_0(t - \tau_{10}), \ldots, x_0(t - \tau_{s0}), u(t))dt.
\]

2. Some Comments

By standard way from 3) it follows the pointwise maximum principle

\[
\psi(t)f(t, x_0(t), x_0(t - \tau_{10}), \ldots, x_0(t - \tau_{s0}), u_0(t)) = \max_{u \in U} \psi(t)f(t, x_0(t), x_0(t - \tau_{10}), \ldots, x_0(t - \tau_{s0}), u), t \in I.
\]

Let \( \tau_{i0} = \tau_{1i}, i = \overline{1, s_1} \) and \( \tau_{i0} = \tau_{2i}, i = \overline{s_1 + 1, s} \) then instead 2) we have

\[
\pi Q_{0r_i} \leq \int_{t_0}^{t_1} \psi(t)f_y[t]x_0(t - \tau_{1i})dt, i = \overline{1, s_1}
\]

and

\[
\pi Q_{0r_i} \geq \int_{t_0}^{t_1} \psi(t)f_y[t]x_0(t - \tau_{2i})dt, i = \overline{s_1 + 1, s}.
\]
If
\[ \text{rank}(Q_{0r_1} \ldots Q_{0r_s} Q_{0z}) = 1 + l \]
then \( \psi(t) \neq 0 \).

3. ON THE EXISTENCE OF AN OPTIMAL ELEMENT

Let \( U \) be compact set.

**Theorem 2.** There exists an optimal element \( w_0 \) if the following conditions hold:

a) \( W_0 \neq \emptyset \);

b) there exists a compact set \( K_0 \subset O \) such that for an arbitrary \( w \in W_0 \)
\[ x(t; w) \in K_0, t \in I; \]
c) the set
\[ \left\{ f(t, x, y_1, \ldots, y_s, u) : u \in U \right\} \]
is convex for all fixed \( (t, x, y_1, \ldots, y_s) \in I \times K_0^{1+s} \).

**Remark 1.** Let \( U \) be convex set and let
\[ f(t, x, y_1, \ldots, y_s, u) = A(t, x, y_1, \ldots, y_s) + B(t, y_1, \ldots, y_s)u \]
then the condition c) holds.

Theorems 1 and 2 are proved by the scheme given [3-5].

**Keywords:** Neutral equation, optimal control, existence.

**AMS Subject Classification:** 34K40, 49J21, 49K21.

**References**


A GENERAL AND SUITABLE METRICS IN FUZZY SPACE

F.I. GURBANOV¹, N.G. MAMEDOVA¹

¹Baku State University, Baku, Azerbaijan
e-mail: fndu@bk.ru

ABSTRACT. The measure of distance between two fuzzy sets is a fundamental tool within fuzzy set theory. However, current distance measures within the literature do not account for the direction of change between fuzzy sets and other features of complex systems, which were modeled with fuzzy sets. In this paper, we highlight this utility and introduce a new opinion of distance measure which takes various properties of complex systems.

Some metric functions and relevant metric space for fuzzy numbers introduced. The common methods of introducing metric functions is described.

Keywords: Metric, fuzzy set, measure, distance, complex system, fuzzy number, space.

AMS Subject Classification: 93C42.

1. INTRODUCTION

It is known that the qualitative and quantitative study of different processes of environment starts with constructing of mathematical models. To improve the adequacy of such models, especially for humanistic systems [3], the fuzzy numbers are widely used in a number of fields. Due to these numbers the uncertainties which appears in complex systems can be effectively modeled in many cases. As a result this leads to effective investigation and management of such systems. Consequently, the study of different properties algebraic constructions, the introduction of metrics and the study of different spaces of such numbers is of great importance for application purposes.

Distance measures for fuzzy sets are an important tool and have been applied to many fields. For example, Bonissone [1] illustrated examples of applying distance measures in decisions analysis and artificial intelligence and Wang and Xing [4] demonstrate distance measures applied to pattern recognition [5], particularly to the problem of classification.

Distance measures that are currently in the literature do not account for the “change in direction” between fuzzy sets and at the same time not sensitive to change properties of complex systems. That is they reveal the distance between two fuzzy sets, but they do not indicate if a fuzzy set is placed to the left or right of another fuzzy set; a concept which will prove useful within computing with Words (CW) and the ranking of fuzzy numbers [2].

In this paper we consider the methods by which its possible to receive suitable metrics for modeling and solving problems. We regard the space of metric functions and search optimal metric function inside spaces [6].
2. Some metric functions and distances

**Definition 1.** [3] A function \( s : [0, 1] \rightarrow [0, 1] \) is a reducing function if \( s \) is increasing and \( s(0) = 0, s(1) = 1 \). We say that \( s \) is a regular reducing function if \( \int_0^1 s(\alpha) d\alpha = \frac{1}{2} \).

**Definition 2.** If \( \mu \) is a fuzzy number with \( \alpha \)-cut representation, \((L(\alpha), R(\alpha))\), and if \( s \) is a reducing function then the value of \( \mu \) is defined by (with respect to \( s \)).

\[
\text{Val} \mu = \int_0^1 s(\alpha)[L(\alpha) + R(\alpha)] d\alpha.
\]

Using the notion of value we can define a pseudo-metric \( d \) for \( F \) by setting:

\[
d(u, v) = |\text{Val} \mu - \text{Val} \nu|
\]

for each \( \mu, \nu \in F \). It is clear from the definition of value that for fuzzy number \( \mu, \nu, \eta \) we have

\[
d(\mu, \mu) = 0, \quad d(\mu, \nu) = d(\nu, \mu),\]

\[
d(\mu, \nu) \leq d(\mu, \eta) + d(\eta, \nu).
\]

It is also clear that there exist distinct fuzzy numbers \( \mu \) and \( \nu \) such that \( d(\mu, \nu) = 0 \), and so \( d \) is not a metric for \( F \).

Let’s introduce 3 metric functions and consider corresponding metric spaces.

Suppose, \( \mu, \nu \in F \), then

\[
supp \mu = [\mu_l, \mu_r], \quad supp \nu = [\nu_l, \nu_r], \quad \text{where supp } \mu = \{x | \mu(x) > 0\},
\]

\[
\mu_l = \{\inf x | x \in supp \mu\},
\]

\[
\mu_r = \{\sup x | x \in supp \mu\}.
\]

**Definition 3.** Fuzzy number \( \mu \) is considered normal number if \( \exists x \in R \), that \( \mu(x) = 1 \), \( c_\mu = \{x \mid \mu(x) = 1\} \).

**Definition 4.** Let \( \mu, \nu \in F \), the Hausdorff metric for this numbers defined as:

\[
D_H(\mu, \nu) = \max\{|\mu_l - \nu_l|, |\mu_r - \nu_r|\}
\]

\[
D_H^1(\mu, \nu) = \int_0^1 D_H(\mu_\alpha, \nu_\alpha) d\alpha, \quad \text{where } \mu_\alpha = \{x | \mu(x) \geq \alpha\}, \quad 0 \leq \alpha \leq 1.
\]

**Definition 5.** Let \( \mu, \nu \in F \) consider the shift fuzzy number\( \mu \) such that its center \( c_\mu \) coincidence with the center of \( c_\nu \) the center of fuzzy number \( \nu \). Lets denote new shifting fuzzy number \( \mu_\nu \). Define the distance by following formula:

\[
D_H^2(\mu, \nu) = \int_{-\infty}^{\infty} |\mu_\nu - \nu| dx, \quad \text{where } \mu_\nu = \mu_\nu(x), \quad \nu = \nu(x).
\]

**Theorem 1.** The metric spaces \( (F, D_H), (F, D_H^1) \) and \( (F, D_H^2) \) are complete and separable. The functions \( D_H, D_H^1 \) and \( D_H^2 \) are metric functions.
3. SPACE OF METRIC FUNCTIONS

In the process of modeling various complex systems it is very important to choose convenient metrics in the set of fuzzy numbers. Unlike crisp sets, in this case we have great opportunities for interpretation, comprehension and definition of different metrics.

The basic idea for introduction of the metrics in this paper is that fuzzy numbers are considered as objects with fixed number of different characteristics. The distance between them is some composition of distances between these characteristics. Every inside metrics play some role in whole distance. The significant and contribution every inside metrics we will estimate with parameter $\alpha_i (1 \leq i \leq n)$.

**Definition 6.** The distance between any fuzzy numbers $\mu$ and $\nu$ ($\mu, \nu \in F$) defined by following formula:

$$d(\mu, \nu) = \sum_{i=1}^{n} \alpha_i d_i(\mu_i, \nu_i),\quad (4)$$

where $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^{n} \alpha_i = 1$.

Here $d_i$ is various metric functions, which are in some sense calculate distance between fuzzy numbers $\mu$, and $\nu$, or between some characteristics of these fuzzy numbers.

If we chose $m$ very significant metric functions $\{d_i\}_{i=1}^{m}$, then by formula (4) we can receive some linear space of metric functions $M_F^m$, where $\{d_i\}_{i=1}^{m}$ are basis functions. By the choosing parameters $\{\alpha_i\}_{i=1}^{n}$ we can construct convinient metric function for considered problem in complex system.

**Theorem 2.** If in formula (4) functions $d_i$ is metric, then the $d(\mu, \nu)$ is metric function.

**Example.** Using metric functions given in (2) and (3) we can merge them and to get fusion metric function:

$$D_H^3(\mu, \nu) = \alpha_1 \int_0^1 D_H(\mu_\alpha, \nu_\alpha) d\alpha + \alpha_2 \int_{-\infty}^{\infty} |\mu_\alpha - \nu_\alpha| d\alpha$$

$$\alpha_1 + \alpha_2 = 1, \quad 0 \leq \alpha_1, \quad \alpha_2 \leq 1.\quad (5)$$

4. CONCLUSION

Thus combining different metric functions as a part of single function we can get more sensitive metric function.

This will allow us to create a more adequate model for complex systems under consideration.

**REFERENCES**

OPTIMIZATION OF MECHANIZED METHODS OF OPERATION TAKING INTO ACCOUNT THE PRODUCTION GAS RATE

T.G. GURBANOVA1, R.S. GURBANOV1

1Scientific-Research Institute Geotechnological Problems of Oil Gas and Chemistry, Baku, Azerbaijan
e-mail: turkanqurbanzade@mail.ru, ramiz.gurbanov@yahoo.com

1. Equation of gas-hydrodynamic processes in the drainage zone of wells

It should be noted that in the drainage zone of fountain, gas-lift and pumping wells, gas-hydrodynamic processes occur of the same nature, depending on the production rate of the reservoir gas. Investigation of these processes has established that in the drainage zone the flow rate and bottom hole pressure are expressed by polynomials of the second order, depending on the production gas rate [1].

\[ Q = aV^2 + bV + c, \]  
\[ P = a_1V^2 + b_1V + c_1, \]

where \( Q \) and \( P \) - are the flow rate and pressure; \( V \) - is the debit of formation gas; \( Q_o \)-the amount of oil, equal (1 - \( \beta \))\( Q \); \( \beta \) - percentage of water cut in well production \( a, b, c, a_1, b_1, c_1 \) - constant coefficients, determined on the basis of well survey data processing. To determine the desired equation, the formation rate of the formation gas from the second-order polynomial of the bottomhole pressure, the substitution in the polynomial of the second order of production and vice versa, we obtain an equation describing the hydrodynamic processes and the indicator dependence of the drainage zone of the wells

\[ Q = \frac{b_1^2}{2a_1} \left( \frac{a}{a_1} - \frac{b}{b_1} \right) \left[ 1 + \sqrt{1 - \frac{4a_1}{b_1^2}(c_1 - P)} \right] - \frac{a}{a_1}(c_1 - P) + c. \]  

Dependencies of production rate, pressure and ratio of constant coefficients of the investigated well [4].

<table>
<thead>
<tr>
<th>Dependencies for production rate and pressure from production gas flow</th>
<th>Ratio of constant coefficients equations (1) and (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>with a linear inflow into the well</td>
<td>with a non-linear inflow into the well</td>
</tr>
<tr>
<td>( Q = -0, 8941 \cdot 10^{-6}V^2 + +0, 01126955V - 13, 8 )</td>
<td>( K = a/a_1 = 9, 038; )</td>
</tr>
<tr>
<td>( P = 0, 98925 \cdot 10^{-7}V^2 - 0, 00153996V + 8, 985 )</td>
<td>( K = b/b_1 = 7, 36; )</td>
</tr>
<tr>
<td>the mean, ( K ) will be 7, 52</td>
<td>( K = (-13, 8)/8, 985 = 7, 52 )</td>
</tr>
<tr>
<td>the mean, ( K ) will be 7, 97</td>
<td>the mean, ( K ) will be 10, 3</td>
</tr>
</tbody>
</table>
2. METHODS FOR STUDYING THE DRAINAGE ZONE OF GASLIFT WELLS

In this paper, theoretical foundations have been developed for the optimization and control of mechanized methods of operation, taking into account the production rate of the reservoir gas [2, 3]. To solve the problem, using the proposed approach, the flow polynomials compiled for the gas measured at the mouth, the supplied working agent and the formation gas will be similar or equal. The rate of formation gas is determined from the dependence of the found theoretical and verified experimentally in a real well.

To solve the problem, using the proposed approach, the flow polynomials compiled for the gas at the mouth of the well $V_{mw}$, the supplied working agent $V_{wa}$, the layer gas $V_{la}$, will be similar or equal. Given the difficulties and inaccuracies in determining the production rate of the reservoir gas, it will be determined from the dependence (1). Dependence of the well production rate on the gas flow rates will be presented according to the dependence (1), and according to the data of the well 198 of the OGMD on May 28 the following dependences

$$Q_{la} = -0.69051 \cdot 10^{-6}V_{la}^2 + 25.35553 \cdot 10^{-3}V_{la} - 47.9137$$

$$Q_{wa} = -3.003 \cdot 10^{-6}V_{wa}^2 + 86.05978 \cdot 10^{-3}V_{wa} - 435$$

$$Q_{mw} = -0.8706 \cdot 10^{-6}V_{mw}^2 + 51.53689 \cdot 10^{-3}V_{mw} - 580.85.$$ 

Processing of experimental well data is given in the table.

Table 2.

<table>
<thead>
<tr>
<th>To naming data</th>
<th>$V_{max}$, $m_3$/day</th>
<th>$V_{opt}$, $m_3$/day</th>
<th>$Q_{max}$, $m_3$/day</th>
<th>$Q_{opt}$, $m_3$/day</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flow rate and formation gas</td>
<td>18,3</td>
<td>8,33</td>
<td>184,85</td>
<td>115,41</td>
</tr>
<tr>
<td>Liquid flow and working agent consumption</td>
<td>14,33</td>
<td>12,04</td>
<td>181,33</td>
<td>165,58</td>
</tr>
<tr>
<td>Flow rate of liquid and gas at the wellhead</td>
<td>29,6</td>
<td>25,83</td>
<td>181,92</td>
<td>169,548</td>
</tr>
</tbody>
</table>

From Table 2. it can be seen that in the optimal and maximum modes the rates of the formation are approximately equal, and the gas flow rates in optimal and maximum modes are located along the gas production axis:

Therefore, the working regime of the reservoir must be at the right (C) of the maximum regime [4]. In order for the bed to work in a stable mode (C), and the gas-lift system will operate in a rational mode (F).

3. PROCESSES WITH LINEAR AND NON-LINEAR INFLOWS INTO THE WELL

The case of nonlinear fluid flow into the well is considered on the basis of the model:

$$P_{la} - P_{wb} = AQ + BQ^2.$$  

From equation (4) it follows that when $P_{wb} = 0$ the debit is equal to the potential production rate of the well

$$P_{la} = A(Q_{pot} - Q) + B(Q_{pot}^2 - Q^2).$$  

From equations (4) and (5) for bottomhole pressure, we obtain [4]

$$P_{wb} = AQ_{pot} + BQ_{pot}^2.$$

Constants of the coefficients A and B, are determined from the dependence (4) on the basis of well research data. With a nonlinear inflow $Q_{pot}$ equals:

$$BQ_{pot}^2 + AQ_{pot} - P_{la} = 0.$$
According to the results of the research of a fountain well conducted by prof. A. Silash determined the constant coefficients A and B and obtained [5].

\[
\Delta P = 0.14Q + 0.001986Q^2. \tag{7}
\]

Using the dependence (7), the potential flow rate of the flowline well

\[
Q_{pot} = -A \pm \sqrt{A^2 + 4BP_{la}} \quad \frac{2B}{2B}
\]

\[
= -0.14 \pm \sqrt{0.14^2 + 4 \times 7.15 \times 0.001986} = -0.14 \pm \sqrt{0.001986 + 0.0556} =
\]

\[
\begin{align*}
&= \frac{-0.14 + \sqrt{0.0756}}{4 \times 10^{-3}} = \frac{-0.14 + 0.27}{4 \times 10^{-3}} = 32.5, \\
Q_{pot} &= 32.5 \text{ m}^3/\text{day}.
\end{align*}
\]

4. Conclusion

1. As a result of the study, the equation of gas-hydrodynamic processes of the drainage zone of oil and gas wells is obtained.

2. The obtained equation can be used for optimization and control of gas-hydrodynamic processes of drainage zone of fountain, gas-lift and pumping wells.

3. Investigation of gaslift wells should be carried out so that the stratum operates in an optimal and stable regime.

![Figure 1](image_url)

Figure 1. For a hypothetical well, a relationship is constructed between the flow rate of a liquid and a gas: 1-for a gas-liquid lift: optimal A and maximum B modes of the elevator; 2-for the productive formation: optimal A1 and maximum B1 formation modes.

Keywords: Gas dynamic processes, drainage zone, well, second-order polynomial, hoist, filter, optimal regime, lifting pipe shoe, production rate, layer gas, rational mode.

AMS Subject Classification: 76N15.

References


ON THE BOUNDARY FUNCTIONAL OF THE RANDOM WALK WITH TWO BARRIERS RELATED TO OPTIMAL CAPACITY OF THE BUFFER STOCK

Z. HANALIOGLU¹, B. GEVER², A. POLADOVA³, T. KHANIYEV³

¹Karabuk University, Department of Actuary and Risk Management, Karabuk, Turkey
²STM Defense Technologies and Trade Inc., Artificial Engineering and Data Fusion, Ankara, Turkey
³TOBB University of Economics and Technology, Department of Industrial Engineering, Ankara, Turkey

e-mail: zulfiyamammadova@karabuk.edu.tr

Abstract. In this study, a boundary functional \( (N) \) of the semi-Markovian random walk \( (X(t)) \) with two special barriers is considered. The boundary functional \( N \) is defined as the first time when the random walk exits from the interval \((-a, a)\). In this study, the boundary functional \( N \) has been investigated under the assumption that the jumps of the random walk are expressed by bilateral exponential distributed random variables. There are significant implementations of the boundary functional \( N \) in the stock control theory. Especially, it is important to investigate numerical characteristics of the boundary functional \( N \) for the finding optimal capacity of buffer stock located between two machines which are working at the same speed. For this reason, the exact expressions for the first three moments of the boundary functional \( N \) are obtained by using basic identity for random walk (Feller (1971)). Next, the exact and approximation expressions for the expected value, variance, standard deviation, variation and skewness coefficients of the boundary functional \( N \) are derived.

Keywords: Random walk with two barriers, Boundary functional, Bilateral exponential distribution, Basic identity for random walk.

AMS Subject Classification: 60G50.

1. Introduction

It is well known that many problems of queuing theory, stock control, reliability, insurance risk, etc., can be expressed by the random walks with two barriers. In the literature, there exists many interesting paper dealing with these problems (see, for instance, Afanasyeva and Bulinskaya, Aliyev and Khaniev, Borovkov, Feller, Gihman and Skorohod, Janseen and van Leeuwaarden, Khaniev and Kucuk, Khaniev and etc., Lotov, and etc.). But most of these studies are generally theoretical and are not helpful enough in solving concrete problems in practice due to the complexity of their mathematical structure. For this reason, in this study a random walk with two barriers is constructed and its an important boundary functional is investigated. The exact and open expressions for the numerical characteristics of the boundary functional are
2. Mathematical construction of the process $X(t)$ and boundary functional $N$

Let $\{\xi_n, \eta_n\}, n \geq 1$ be sequence of a random pairs defined on some probability space $\{\Omega, F, P\}$, such that pairs are independent and identically distributed. Moreover, $\xi_n$ takes only positive values, $\eta_n$ takes both positive and negative values. Suppose that the random variables $\xi_n$ and $\eta_n$ are independent from each other and their distributions are known. Let their distribution functions be denoted by $\Phi(t)$ and $F(x)$, respectively. So,

$$\Phi(t) \equiv P\{\xi_1 \leq t\}, \quad F(x) \equiv P\{\eta_1 \leq x\}, \quad t \geq 0, x \in R.$$

Define the renewal sequence $\{T_n\}$ and random walk $\{S_n\}$ as follows:

$$T_0 = S_0 = 0; \quad T_n = \sum_{i=1}^{n} \xi_i; \quad S_n = \sum_{i=1}^{n} \eta_i, \quad n \geq 1,$$

and define a sequence of integer-valued random variables $\{N_n\}, n \geq 0$ as

$$N_0 = 0; \quad N_1 = N = \inf\{k \geq 1 : S_k \notin (-a, a)\}$$

$$N_{n+1} = \inf\{k \geq N_n + 1 : S_k - S_{N_n} \notin (-a, a)\}, \quad n \geq 1.$$

Put $\tau_0 = 0, \quad \tau_1 = T_N = \sum_{i=1}^{N} \xi_i; \quad \tau_n = \sum_{i=1}^{N_n} \xi_i$ and define $\nu(t)$ as follows:

$$\nu(t) = \max\{n \geq 0 : T_n \leq t\}, \quad t > 0.$$

We can now construct the desired stochastic process $X(t)$ as follows:

$$X(t) = S_{\nu(t)} - S_{\nu+1}, \quad \tau_n \leq t \leq \tau_{n+1}, \quad n \leq 0.$$  

The process $X(t)$ is interpreted as a semi- Markovian random walk with two barriers. In this study, it is assumed that the random variable $\eta_1$ has a bilateral exponential distribution with parameter $\lambda = 1$. In other words, probability density function of $\eta_1$ is as $f_{\eta_1}(x) = \frac{1}{2}e^{-|x|}, \quad x \in R$.

Moreover, the boundary functional $N = \inf\{k \geq 1 : S_k \notin (-a, a)\}$ interpreted as the first time when the random walk exits from the interval $(-a, a)$. In terms of buffer stock the random variable $N$ interpreted as the number of loading and dumping which is necessary for completely filling and discharging of the stock. The goal of this study is to investigate the numerical characteristics of the random variable $N$. Hence, using basic identity for random walk (Feller (1971)), the expected value, variance, standard deviation, coefficient of variation and skewness coefficient of the random variable $N$ are obtained.

3. Investigation of the boundary functional $N$

Feller (1971) has shown that, when $\eta_n$ has bilateral exponential distribution function, boundary functionals $N$ and $S_N$ are independent from each other and by using basic identity for random walk, characteristic function of boundary functional $S_N$ and moment generating function of boundary functional $N$ are obtained as follows:

$$\Phi_{S_N}(u) \equiv E\{\exp(iuS_N)\} = \frac{\exp(iau)}{2(1 - iu)} + \frac{\exp(-iau)}{2(1 + iu)}, \quad u \in R,$$

$$\Psi_N(z) \equiv E(z^N) = \left[\frac{\exp(-a\sqrt{1 - z})}{2(1 + \sqrt{1 - z})} + \frac{\exp(a\sqrt{1 - z})}{2(1 - \sqrt{1 - z})}\right]^{-1}, \quad |z| \leq 1.$$
Using exact expression for $\Psi_N(z)$, the exact expressions for the first three moments of the boundary functional $N$ can be written as follows:

$$
E(N) = \frac{1}{2}a^2 + a + 1 \quad (Feller\,1971);
$$
$$
E(N^2) = \frac{5}{12}a^4 + \frac{5}{3}a^3 + \frac{7}{2}a^2 + 3a + 1;
$$
$$
E(N^3) = \frac{61}{120}a^6 + \frac{61}{20}a^5 + \frac{19}{2}a^4 + 16a^3 + \frac{31}{2}a^2 + 7a + 1.
$$

Therefore, explicit and approximated expressions for the numerical characteristics of the boundary functional $N$ are derived as follows:

$$
Var(N) = \frac{1}{6}a^4 + \frac{2}{3}a^3 + \frac{3}{2}a^2 + a \approx \frac{1}{6}a^4 + \frac{2}{3}a^3 + \frac{3}{2}a^2;
$$
$$
\sigma_N = \sqrt{Var(N)} = \sqrt{\frac{1}{6}a^4 + \frac{2}{3}a^3 + \frac{3}{2}a^2 + a} \approx \sqrt{\frac{6}{3}} \left( 1 + \frac{2}{a} + \frac{5}{2a^2} \right);
$$
$$
CV(N) = \frac{\sigma_N}{E(N)} = \frac{\sqrt{\frac{1}{6}a^4 + \frac{2}{3}a^3 + \frac{3}{2}a^2 + a}}{\frac{1}{2}a^2 + a + 1} \approx \frac{\sqrt{6}}{3} \left( 1 + \frac{3}{a^2} \right);
$$
$$
E(N - E(N))^3 \approx \frac{2}{15}a^6 \left( 1 + \frac{6}{a} + \frac{75}{4a^2} \right) ; \quad \sigma_3^3 = \frac{1}{8}a^6 \left( 1 + \frac{6}{a} + \frac{72}{4a^2} \right);
$$
$$
\gamma_3 = \frac{E(N - E(N))^3}{\sigma_3^3} \approx \frac{16}{15} \left( 1 + \frac{3}{4a^2} \right).
$$

Remark. Obtained results play a significant role to calculate the optimal capacity of the buffer stock located between two machines.

References

ESTIMATION OF IMPACT OF INNOVATIONS ON THE QUALITY OF TERTIARY EDUCATION

Y.H. HASANLI¹, S.A. SHABANOV¹

¹ Azerbaijan State University of Economics, Baku, Azerbaijan
e-mail: yadulla.hasanli@unec.edu.az, sardar.shabanov@unec.edu.az

In order to estimate the impact of innovations on the quality of education it is important to have the indicators that those ones are able to characterize these definitions. It is known that the indicators impacting to quality of education can be divided into two parts: measurable and immeasurable. For example, how to characterize the education system of any country? There is no doubt that international comparisons are necessary here [6,7]. On the other hand, the education itself consists of different stages. It is known that education has both quantitative and qualitative indicators. For example, quantitative indicators include the level of literacy of the country’s population, the number of preschool educational institutions, the number of secondary schools, and the number of teachers and so on. However, besides these indicators, in recent years, special attention has been paid to the quality of education. For example, the student’s average score (at the pupil level) in the math, average scores on mathematics and natural sciences in the international PISA competition, “good” teacher, national tradition, culture, etc. [2,5,6].

In this study, we will use only the criterion of the number of journal articles falling to highly indexed scientific bases that allow us to describe the quality of tertiary education in the country to a certain extent. This criterion was first used in the article [3] published in 2013. It should also be noted that the article must concern only the fields of basic science and engineering, but not in any other area. This is due to the fact that these studies show that just these fields have a higher impact on the quality of tertiary education from a statistical point of view. That is to say, the impact of these fields is stronger than the impact of economics and business. So, in order to characterize the quality of tertiary education in the country, we will use the indicators of the number of articles concerning the fields of basic science and engineering and indexed by Web of Science. The most important feature of this indicator is its accessibility and one-dimensionality.

Now let’s pay attention to innovation. A number of innovation indicators are used in the World Bank database [4]. We can show the followings as examples:

1. High-technology export (% of manufactured export);
2. High-technology export (current US $);
3. Number of technical staff working in Research and Development (per million people);
4. Number of researchers in Research and Development (per million people);
5. Trademark applications;
6. Patent applications;
7. Share of Research and Development expenses in GDP (%);
8. Intellectual property export (US $);

The analysis was conducted by constructing the econometric models of impact of innovations on the quality of tertiary education for Azerbaijan, Germany, Austria, Russia, Kazakhstan, Iran.
and Turkey based on relevant indicators covering 2000-2017. Econometric models [1] were implemented through the Eviews 9 program. We took the number of articles as a dependent variable and we set up various regression equations and run correlation and regression analysis. The impact of the innovation on the number of articles that we have investigated has been statistically significant for only two indicators: 1) Intellectual property export (US $); 2) Intellectual property import (US $). That’s why we removed the remaining indicators besides the two ones from our research. Some information for Azerbaijan is shown in Table 1.

<table>
<thead>
<tr>
<th>Year</th>
<th>ImportIP (US $)</th>
<th>SciTechArticles</th>
</tr>
</thead>
<tbody>
<tr>
<td>2002</td>
<td>2 050 000</td>
<td>216.6</td>
</tr>
<tr>
<td>2003</td>
<td>144 000</td>
<td>245.6</td>
</tr>
<tr>
<td>2004</td>
<td>N/A</td>
<td>327.5</td>
</tr>
<tr>
<td>2005</td>
<td>46 000</td>
<td>329.8</td>
</tr>
<tr>
<td>2006</td>
<td>2 281 000</td>
<td>280</td>
</tr>
<tr>
<td>2007</td>
<td>4 697 000</td>
<td>408.6</td>
</tr>
</tbody>
</table>


The regression equation of the dependence of the number of articles on the import of intellectual property for Azerbaijan was found as follows:

$$\ln(\text{SciTechArticles}) = 1.252816 - 1.118388 (\ln(\text{ImportIP}))^2$$ (1)

Here, $\ln(\text{SciTechArticles})$ – the natural logarithm of the number of articles concerning the fields of basic science and engineering and entered in influential base; $\ln(\text{ImportIP})$ - the natural logarithm of intellectual property imports.

<table>
<thead>
<tr>
<th>Year</th>
<th>ImportIP (US $)</th>
<th>SciTechArticles</th>
</tr>
</thead>
<tbody>
<tr>
<td>2008</td>
<td>4 827 000</td>
<td>470.5</td>
</tr>
<tr>
<td>2009</td>
<td>19 229 000</td>
<td>617</td>
</tr>
<tr>
<td>2010</td>
<td>16 531 000</td>
<td>618.7</td>
</tr>
<tr>
<td>2011</td>
<td>18 423 000</td>
<td>655.8</td>
</tr>
<tr>
<td>2012</td>
<td>28 180 000</td>
<td>683.1</td>
</tr>
<tr>
<td>2013</td>
<td>N/A</td>
<td>481.8</td>
</tr>
</tbody>
</table>

Source: the authors’ calculations through Eviews.

All the statistical characteristics of this equation are satisfactory. Thus, the standard errors of the parameters are much smaller than the value of the parameter and it shows that the values found by the least squares method are highly reliable (t-test). Being Durbin-Watson statistics close to 2 (DW = 2.1181) shows that there is no first difference autocorrelation of the residuals and this is desirable. The value of adjusted $R^2$ is close to one ($R^2_{adj}$) indicates that the model has high levels of approximation. In other words, the change in the number of articles can be explained by the 93.3% change in the volume of imported intellectual property. It is obvious from the equation that the form of dependency is a square parabola oriented upward. Before the value of his ordinate decreases then it increases.

The computation shows that theoretically the minimum value is obtained in the case that the imports of intellectual property is equivalent to $241,822. That is to say after the import of intellectual property exceeds $ 242,000, the number of articles concerning the fields of basic science and engineering and entered in influential base starts rising, otherwise the decline is observed.

If we compare this obtained result with the official statistics, we will see that although the value was not in the database, such value occurred in the case wherein the closest value was 280 in 2006, while the amount of imported intellectual property was $ 1281,000. The value of average elasticity coefficient for Azerbaijan was computed as $E = 0.385$.

This means that if the value of imported intellectual property in Azerbaijan increases by 1% compared to the average, then the number of articles concerning the fields of basic science and engineering and entered in influential base will increase by approximately 0.385% compared to the average.

The econometric modeling results for Germany, Austria, Russia, Kazakhstan, Iran and Turkey are as follows:
1% increase in intellectual property exports for Germany increases the number of articles by 0.26%, but increase in imports by 1%, increases the number of articles by 0.12%. The impact of intellectual property exports on the number of articles is about twice as much as imports. The change in explanatory variables explains 95.81% of the change in response variable.

For Austria it was determined that neither export nor import of intellectual property had significant impact on the number of articles. The econometric equation is constructed on the share of research and development in GDP. It was found that if the share of research and development in GDP changes in Austria by 1% compared to the average, then the number of articles compared to the average will change by 1.332% too.

The econometric equation determining the dependence of the number of articles on imported intellectual property was constructed for Russia. The average elasticity value was found as 0.124. This means that change in intellectual property imports by 1% will change the number of articles approximately 0.12%.

It was found out that the number of articles for Kazakhstan does not depend on the intellectual property export. The elasticity coefficient of the dependence of the article number on import for Kazakhstan was calculated: $9.390 \times 10^{-9}$, it is practically zero. That is to say the number of articles for Kazakhstan is not elastic in relation to the volume of intellectual property imports.

Since there was no other information in appropriate source about the two other indicators that interested us except for the number of articles, it was impossible to establish a regression equation for the 2000-2016 periods for case of Iran. However, the dynamics of the article show that science has rapidly evolved over the same years and the average annual growth rate of articles was 24.18%.

The econometric equation determining the dependence of the number of articles on imported intellectual property was constructed for Turkey and the appropriate average elasticity value was computed as $E = 0.761$. This shows that 1% increase (decrease) in intellectual property imports compared to the average for Turkey will increase (decrease) the number of articles by 0.76%. Note that coefficients of regression equation for Turkey are statistically significant with 1% risk. That is also the same for other countries those we have explored. The computations show that the coefficients for Germany were statistically significant at 5% risk.

Finally, it should be noted that with considering the special role of the export and import of intellectual property for the development of science and the improvement of the quality of tertiary education, Azerbaijan either should pay particular attention to the fields of basic science and engineering or increase their efforts on both intellectual property exports and imports by developing the fields of knowledge-intensive processing industries.

Keywords: Education, quality, intellectual property, econometrics, elasticity.

AMS Subject Classification: 62P20.

References
MATHEMATICAL MODEL OF MASS-EXCHANGE IN MEDIUM FRACTAL STRUCTURES

A.B. HASANOV¹, E.N. SADIGOV²

¹Baku State University, Baku, Azerbaijan
²Institute of Control Systems of ANAS, Baku, Azerbaijan
e-mail: dahi57@rambler.ru

ABSTRACT. A mathematical model of the problem of fluid filtration in layered porous media was proposed based on the analysis of the factors that take into account the filtration laws and the fractal characteristics of the porous media, using the formalized separation differentiation and integration. In order to model the geometrical position of the pores it is more advantageous to use from the dimensional spaces [1]. Increasing the mathematical model’s adequacy, using stereotypical stochastic displacement of soil-ground layer for more real environments, and formalism of contouring differentiation, taking into account the fractal structure of porous media [1].

Keywords: Fractal structure, viscoelastic environment, variable structure medium, filtration problems.

AMS Subject Classification: 74F10.

1. INTRODUCTION

Despite the serious and significant investigations, the issue of establishing adequate models of unstable physical processes is still relevant. Some features of the mathematical modeling of non-local transfer processes should be studied when modeling the ground-ground system as fractal-structured porosity environments. It is crucial to investigate unstable filtration processes in porous media, especially where the pores are complex topologies.

2. FORMULATION OF THE PROBLEM

Taking into account the stochastic displacement of the soil-ground layer and the fractal character of the porous media and using the known expressions for the fractional differentiation one can obtain

\[ D_{0t}^a \varphi = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(-a)} \int_0^t \frac{\varphi(\tau)d\tau}{(t-\tau)^{a+1}}, & a < 0 \\
\frac{\partial^{[a]+1}}{\partial t^{[a]+1}} D_{0t}^{a-[a]-1} \varphi, & a > 0.
\end{array} \right. \]

Then

\[ h_1 = k (1 - a) D_{0t}^{a-1} \frac{\partial h(x, y, \tau)}{\partial \tau}, \]
where $D_{\alpha}^{\mathcal{X}-1}$ is an operator of fractional differentiation. The use of the fractional differentiation operator of M. Caputo sense gives

$$D^{-}\alpha_{0}^{t} \frac{\partial h(x, y, \tau)}{\partial \tau}, \quad 0 < a \leq 1.$$ 

Comparing the obtained expressions we get

$$h_{t} = k (1 - a) \partial_{\alpha_{0}^{t}} h(x, y, t)$$

or

$$k (1 - a) \partial_{\alpha_{0}^{t}} (m h(x, y, t)) = \frac{\partial}{\partial x} \left( K \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( K \frac{\partial h}{\partial y} \right) - \frac{k_{0}}{k d_{0}} K + f_{0}.$$ 

This equation is called a generalized Bussinesk equation for time. One can easily check that

$$\lim_{a \to 1} \partial_{\alpha_{0}^{t}} h(x, y, t) = \frac{\partial h(x, y, t)}{\partial \tau}.$$ 

So, for the case

$$k = \frac{1}{(1 - a)}$$

$\alpha \to 1$ this equation turns to the classical Bussinesk equation

$$\frac{\partial h}{\partial t} = a \left( \frac{\partial^{2} h}{\partial x^{2}} + \frac{\partial^{2} h}{\partial y^{2}} \right) + f (x, y, t),$$

More proper form is as follows

$$\frac{\partial (m h)}{\partial t} = \frac{\partial}{\partial x} \left( K \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( K \frac{\partial h}{\partial y} \right) - \frac{k_{0}}{k d_{0}} K + f_{0},$$

$$f_{0} = -\frac{k_{0}}{d_{0}} (h_{0} - H_{0}) + w.$$ 

The linearized solution method of this nonlinear equation considering the the physical considerations, is given above. Taking into account the fractal structure, we get a new expression for the study of the level variations of groundwater

$$h_{t} = k \int_{0}^{t} (t - \tau)^{-a} \frac{\partial h(x, y, \tau)}{\partial \tau} d\tau.$$ 

From this

$$a = \frac{k h_{\mathcal{X}P}}{m}, \quad f = \frac{f_{0}}{m} - \frac{k_{0} h_{\mathcal{X}P}}{m d_{0}}.$$ 

Despite the serious and significant investigations by researchers, the problem of establishing adequate models of unstable physical processes is still relevant. In the porous medium, the filtration equation can generally be written as follows [1,4]:

$$\frac{\partial (m \rho)}{\partial t} = \text{div} \left( \rho \overrightarrow{V} \right), \quad \overrightarrow{V} = -\frac{k}{\mu} \text{grad} \rho. \quad (1)$$

The equations of state $\rho = \rho (P, T)$ and porosity $m = m (P, T) = m (\rho)$ also should be added to this equation. Though solutions to this system have been found in various modifications and applied, many of them do not take into the consideration the "memory" effect of the medium, the spatial correlation of the pores, and generally the non-linarity of the system inadequacy. Let us generalize system (1) for the implementation of the fractal order derivative

$$l_{0} \partial_{\alpha_{0}^{t}} (m \rho) + t_{0} D_{\alpha_{0}^{t}}^{\beta} (\rho V (\xi)) = 0,$$

$$V (\xi) = -\frac{k}{l_{0} \mu} D_{\alpha_{0}^{t}}^{\beta} P (\xi, \tau). \quad (2)$$
Here \( D_0^\beta f(\xi, t) = \frac{\sec(0.5\beta)}{2(2-\beta)} \cdot \frac{\partial}{\partial \xi} \int_{-\infty}^\infty \frac{f(\sigma, \tau)}{\xi - \sigma^\beta} d\sigma^\beta \) is Ritz-Weil derivative. System (2) may be reduced to the following equation

\[
l_0 \frac{\partial}{\partial t} (\partial^\beta \rho) - t_0 D_0^\beta \left( \frac{k \rho}{\mu} \rho D_0^\gamma P(\xi, \tau) \right) = 0. \tag{3}
\]

Equation (3) is a closed loop system together with the state equation \( \rho = \rho(P, T) \). When the porosity coefficient is constant in the medium, for the non-compressed fluid equation (3) turns

\[
\frac{\partial^\beta}{\partial t^\beta} P(\xi, \tau) = B_0 \frac{D_0^\beta}{D_0^\mu} \left\{ D_0^\gamma P(\xi, \tau) \right\}, \tag{4}
\]

where \( B_0 = \frac{bk \ell_0}{m \ell_0 \mu} \). As an initial condition is taken \( P(\xi, 0) = B(\xi) \). Here \( P \) is a fluid inside layer pressure, \( \mu \) is the absolute viscosity of the liquid, \( k \) - conductivity of the medium, \( \beta \)-fluid volume elasticity module. For the case \( \beta = \gamma = 1 \) equation (4) can be rewritten as

\[
\frac{\partial^\beta}{\partial t^\beta} P(\xi, \tau) = B_0 \frac{\partial^2}{\partial \xi^2} P(\xi, \tau). \tag{5}
\]

The general solution of this equation satisfying initial condition \( P(\xi, 0) = P_0 = \text{const} \) indeed is

\[
P(x, t) = P_0 \int_{-\infty}^\infty e^{-ik\xi}.E_{\alpha,1}(-B_0k^2t^\alpha)dk. \tag{6}
\]

If to take \( \alpha = 1 \) in expression (6) then we get the solution known from literature

\[
P(\xi, \tau) = \frac{P_0}{\sqrt{4\pi A\tau}} \int_{-\infty}^\infty \exp \left(-\frac{(\xi - \xi')^2}{4A\tau} \right) d\xi'.
\]

For \( \alpha = 0.5; \alpha = 0.25 \) \( E_{\alpha,1}(\cdot) \) is Mittag-Leffler function. This function can also be presented in the form

\[
E_{1,\frac{1}{2}}(-\sqrt{z}) = \frac{1}{\sqrt{\pi}} - \sqrt{z} \cdot e^z \left(1 - erf \left(\sqrt{z}\right) \right),
\]

\[
E_{1,\frac{1}{4}}(z) = \frac{1}{\left(\frac{1}{4}\right)!} \cdot F_1(1; \frac{1}{4}; z) + \frac{\sqrt{z}}{\left(\frac{3}{4}\right)!} F_1(1; \frac{3}{4}; z) - z^{\frac{3}{2}} \cdot e^z \cdot \left(1 + erf \left(\sqrt{z}\right) \right) - \frac{\sqrt{z}}{\sqrt{\pi}}.
\tag{7}
\]

In the particular case when \( B(\xi) = \delta(\xi) \) expression (6) takes the

\[
P(\xi, \tau) = \frac{1}{\pi} \cdot \int_0^\infty E_{\alpha,1}(-B_0\tau^\alpha k^2)\cos(k\xi) dk.
\]

From this solution taking \( \alpha = 1 \) we obtain the known expression

\[
P(\xi, \tau) = \frac{1}{\sqrt{4\pi B_0^2\tau}} \cdot \exp \left(-\frac{\xi^2}{4B_0^2\tau} \right). \]

Fractional differential is a new approach to the study of non-local processes in space and time and allows one to determine the most important quantitative characteristics of these processes.

**References**


NUMERICAL ALGORITHMS FOR SOLVING THE INVERSE PROBLEM

N.SH. HUSEYNOVA\textsuperscript{1}, M.SH. ORUCOVA\textsuperscript{2}, N.A. SAFAROVA\textsuperscript{1}, N.S. HAJIYEVA\textsuperscript{1}, L.A. RUSTAMOVA\textsuperscript{3}

\textsuperscript{1}Institute of Applied Mathematics, BSU, Baku, Azerbaijan
\textsuperscript{2}Azerbaijan State Economic University, Baku, Azerbaijan
\textsuperscript{3}Baku Satate University, Baku, Azerbaijan
e-mail: nargiz_huseynova@yahoo.com, morucova@mail.ru, narchis2003@yahoo.com

Abstract. In this work the wave equation is analytically solved in the variational form and for the gradient of the functional the analytical expression is found. Also solving the inverse problem with respect to the potential the analytic expression for the optimal potential is obtained.

Keywords: Inverse problem, variation method, optimal potential.

AMS Subject Classification: 31A25, 34A55, 65L09.

1. Introduction

It is known that the motion of a particle in a central field is described by the equation \cite{1-3, 6}
\begin{equation}
-a\frac{d}{dr}\left(\frac{dR}{dr}\right) + bR + q(r)R = ER.
\end{equation}

Here $a > 0$ and $b$ are given numbers and $q(r)$ is the energy of interaction. Multiplying this equation by $r^2$ and denoting $Q(r) = b + q(r)r^2,$

we obtain
\begin{equation}
-a\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + Q(r)R = Er^2R.
\end{equation}

The analytical solution of the equation (2) for different potentials is very interesting. But it is not always possible to obtain the analytical solutions \cite{7}.

In addition, the solution of equation (2), finding of the potential $Q(r)$ with respect to the energy eigenvalues, i.e., solution of the inverse problem is also very interesting.

Assume that $R(r_0) = z_0,$ $R(r_1) = z_1,$ $R(r_2) = z_2,$ ..., $R(r_n) = z_n,$

\begin{equation}
0 < r_0 < r_1 < ... < r_n; \quad n \geq 2.
\end{equation}

We consider the equation (2) on the interval $[r_1, r_n].$ In the work the primary aim is finding the potential $Q(r)$ in the interval $[r_1, r_n].$ We also need to show that the solution of the equation (2) $R(r)$- function satisfies the equation (3).

We will assume that the solution of the equation (2) is $R(r)$ and the condition $\int_0^\infty R(r)dr < +\infty$ is satisfied. Then the solution of the inverse problem is finding the potential $Q(r)$, which it is necessary that by $r \geq 0$, the function $Q(r)$ would be continuously differentiable.
Now we will find the minimum of the following functional

\[ J(Q) = \sum_{i=1}^{n-1} [R(r_i) - z_i]^2 \rightarrow \min, \]  

(4)

from the equation (2) we obtain the following conditions:

\[ R(r_0) = z_0, \quad R(r_n) = z_n. \]

(5)

We suppose that the function \( \psi = \psi(r) \) is solution of the following equation:

\[ -a \frac{d}{dr} \left( r^2 \frac{d\psi}{dr} \right) + Q(r)\psi - Er^2 \psi = -2 \sum_{i=1}^{n-1} [R(r) - z_i]\delta(r - r_i), \]

(6)

**Theorem 1.** Functional (3) is differentiable and its gradient [8] is given by the formula

\[ J'(Q) = \psi R. \]

Theorem 1 allows us to determine the optimal potential analytically.

**Theorem 2.** Let \( Q^* = Q^*(r) \) be the optimal potential for the problem (2), (4), (5). Then for any \( Q = Q(r) \subset U \) the relations

\[ U = \{ Q = Q(r) \in L_2(r_0, r_n) : Q_0 \leq Q(r) \leq Q_1, \ \forall r \in [r_0, r_n] \}. \]  

(8)

are true. Here \( 0 \leq Q_0 < Q_1 \) are given numbers and \( R^* = R^*(r), \psi^* = \psi^*(r) \) solutions of a problem (2), (5) at \( Q = Q(r) \).

\[ \psi(r_0) = 0, \quad \psi(r_n) = 0. \]

(9)

2. An algorithm for solving the inverse problem

1. Consider the arbitrary initial potential \( Q_0 \leq Q(r) \in U \).
2. Found the solution of the equations [4, 5] (2), (5) with the potential \( Q_0(r) \), denote this solution \( R_0 = R_0(r) \).
3. Substituting the solution \( R_0 = R_0(r) \) to the sweep problem (6), (9), solving this problem we find the function \( \psi_0 = \psi_0(r) \).
4. Using the solutions \( R_0 = R_0(r) \) and \( \psi_0 = \psi_0(r) \), we found the gradient of the functional (4).
5. Minimize the linear functional

\[ I_0(Q) = \int_{r_0}^{r_n} \psi_0(r)R_0(r)Q(r)dr \rightarrow \min \]

(10)

in the set \( U \) and find the helper function \( Q_0 = Q_0(r) \). The new potential is constructed as follows:

\[ Q_1(r) = \alpha Q_0(r) + (1 - \alpha)Q_0(r), \quad 0 \leq \alpha \leq 1. \]

6. The accuracy criterion is checked. It may be either such

\[ \max_{r_0 \leq r \leq r_n} |Q_1(r) - Q_0(r)| < \varepsilon, \]

or such \( |J(Q_1) - J(Q_0)| < \varepsilon \). In the 6th step the parameter \( \alpha \) should be chosen thus that the obtained new values of functional with corresponding \( \alpha \) were smaller than previous one \( J(Q_{k+1}) \leq J(Q_k) \) or \( J(\alpha Q_k + (1 - \alpha)Q_k) \leq J(Q) \).

These conditions are called the monotonicity conditions. From the monotonicity conditions can be seen that finding the parameter \( \alpha \) from the condition

\[ J(\alpha Q_k + (1 - \alpha)Q_k) \rightarrow \min, \quad 0 \leq \alpha \leq 1 \]

is advantaged.
However, finding $\alpha$ from these conditions creates additional difficulties. Therefore it is important to give another method, which is important from a practical point of view.

We assume $\alpha = \frac{1}{2}$ and check the monotonicity condition. If the monotonicity condition is satisfied, then the corresponding $\alpha$ iteration is continued. Otherwise, assuming $\alpha = \frac{1}{4}, \frac{1}{8}, \ldots$ the monotonicity condition is checked.

Another way to give iteration formula for each $\alpha$. We can write such as

$$\alpha_k \geq 0, \quad \alpha_k \to 0, \quad \sum_{k=1}^{\infty} \alpha_k = \infty.$$ 

For example we can take $\alpha$ as $\alpha_k = \frac{1}{k+1}$.

Now let’s pay attention to the algorithm’s doing different operations. As seen from second and third processes in each iteration either basic (2), (5) problem or addition (6), (7) problems must be solved. It is not always possible to do it on analytic form so it is convenient to do it by the numerical method. If delta function has entered to the (6), (7) problems, it solution require special approximation. However for solving problems (2), (5) and (6), (7) modern programs such as MATLAB can be used.

From the algorithm can be seen that on each $5^{th}$ step of iteration the linear functional is minimized in the set $\mathcal{U}$. The set $\mathcal{U}$ has a simple structure and it is solution doesn’t create the difficulties. Therefore, the functional (10) is discredited and is operated within constraints to the linear function of minimization, in other words is reduced to the linear programming problem.

3. Conclusion

Let $Q^\ast = Q^\ast(r)$ by the optimal potential for the problem (2), (3), (4). Then

$$Q^\ast(r) = \begin{cases} Q_0, & \text{if } \psi^\ast(r) > 0, \\ Q_1, & \text{if } \psi(r) < 0. \end{cases}$$

In the case $\psi^\ast(r) = 0$, the potential can be chosen arbitrarily.

References


DEFINITION AND CLASSIFICATION OF VARIABLES THAT FORM EVALUATION METHOD FOR DRIVE MECHANISM DECISION MAKING GUIDE FOR IN-PIPE INSPECTION ROBOTS IN OIL PIPELINES

BAHADUR IBRAHIMOV¹, ABBAS ALILI²

¹Azerbaijan Robotics and Automation Society, Baku, Azerbaijan
²Process Automation Engineering department, Baku, Azerbaijan
e-mail: bahaduribrahimov@gmail.com, abbas.alili@bhos.edu.az

Abstract. The purpose of this article is to analyze and evaluate in-pipe inspection robots in terms of developing and offering an Open Innovation Standardization Tool to transform enormous researches of petroleum industry related autonomous solutions to a common platform. Framework of this work allows to inspect and investigate main robot-based technological innovations of petroleum industry, focus on in-pipe inspection robot’s locomotion and developments, highlight the variables and evaluate constraints to create the assessment criteria. Nevertheless, the work aim remains to create guidelines of effective, cost efficient and favorable evaluation and assessment tool to follow solutions which fulfills autonomous system needs in in-pipe inspection of petroleum exploration and production industry. Robotics has a potential to contribute significant benefits for offshore and onshore petroleum industry for the next decades in terms of growth in profits, safety and production capacity. But in fact nowadays it has the problem of missing wide commercial availability [1]. Which in turn, means that application of robotics is more expensive in some cases for the industry that it should be or the development period is complex. Nevertheless, this potential is barely used due to economic, organizational and social barriers. There have been various researches, developments and applications of pipeline inspection robots carried out. It is important to know and analyze the variables for a development in order to state the borderlines of design, to formulate the requirements and have a better planning and evaluation of the developing robot. On the other hand, variables are important part of the planning and supporting phases since in locomotion the parameters are determining the dynamic behavior of robots. In this research, the parameters are defined as variables of locomotion, classified first time in this field and evaluated as shown in following sections.

Keywords: Inspection, robots, in-pipe, petroleum industry.

AMS Subject Classification: 70B13.

1. VARIABLES OF LOCOMOTION

Out of literature reviews and other relevant information collection activities that follows general and specific variable set has been used. Those variables and definitions are explained in following subchapters. However, as a novel approach, two types of classification of the robotic variables are carried out.

Variables are considered due to pipeline topography, inspection activities and robot characteristics. In the first place, the velocity of the robot is considered in almost all projects and it is used in robotic developments, specified as V. Since the continuous delivery is the main goal of the pipeline transportation, the velocity of robots should be as high as possible whilst it should also enable the inspection to be carried in intended accuracy. Velocity of the robot (V) also means how fast the data collection is conducted during the inspection task. Other important
environmental variables to be considered are the pressure inside the pipeline (P), temperature (C), density of the fluid (\(f\)), mass of the fluid (m) and the velocity flow of the fluid inside the pipeline (\(f\)), as well as the pipeline length (L) to be inspected which affects the movement planning of passive locomotion robots and working environment of other types of robots.

The other variable is identified as the weight use factor of a robot (\(I_g\)), denoting the capability of the robot and drive mechanism in carrying the payload (m) of the robot, which signifies that a heavier robot needs more power supply (PS) to climb greater angles between the ground and the pipe wall (\(O = \text{slope angle of the pipeline to ground}\)) for slipping down prevention [4,5].

Radius of the pipeline (R) and Robots radius (\(R_R\)) (can be diagonal length in different shaped robots than cylindrical) should be considered to be co-operative with each other, while robot also should coop with different forces (\(F = \text{sum of all forces}\)) such as tractive force (\(F_t\)), actuator’s torque (T) which creates thrust force (\(F_d\)) also considering the intended or actual movement direction of the robot (M) and the wheel angle to the pipeline base (\(\theta\)); friction force (F) and gravity support forces (N) such as gravity force (N_1). However, the forces are not only measured with the sensors, they could be also measured by the analysis of positions of joints and strings especially in DCAMs. [4], [5]

Adhesion coefficient and material specifications (K) of the pipeline affecting the motion planning and overcoming or coping with the forces also should be considered.

One of the challenges in in-pipe inspection is the overcoming obstacle (\(O_b\)) such as accumulations or bends inside the pipeline. Here, height (ho) and weight of the obstacle (who) define the obstacle variables. For bends and curves, curve angle and curve radius (\(r_x\)) should be revealed or estimated. To overcome bends and curves, a robot’s connections should have flexibility (\(F_l\)), stiffness (St) and degrees of freedom (B) considered for all of robot’s joints (\(#B = \text{number of bodies / universal joints}\)). For example, [2] reviewed a lot of different robotic locomotion approaches and ended up with a unit based wheel robot which is suitable for in-pipe inspection and can overcome branches and elbows depending on the number of units.

Safety and accessibility of the robot (S) as well as communication (Com) of the robot is a variable for operators to consider performing an inspection task. [6]

2. Classification of variables

This study classifies the variables in two ways. Through the first classification, the various parameters are considered as Primary (or main) and Specific (or secondary) variables. Main variables are most commonly mentioned and evaluated variables, such as the velocity, pressure, obstacles, radius and other elements. For specific variables, they tend to vary due to the intention of usage of a robot and other settings required therein. In other words, primary variables for any robot type are environmental specifications, which will be in play when the robot is in operation. For specific variables, explicit parameters are defined which can be specified during the development of a robot such as joints, movement directions, material specifications and other specific aspects.

The second classification method is determined precisely owing to the pre-defined constraints. Variables of in-pipe inspection can either be environmental or robotic variables or those occurring when a robot is interacting with surrounding environment which is the inner part of the pipe where oil flows. At the same time, there can be some specific variables that are not found in any of these categories. Based on the above classification we come up with the following variables:

3. Assessment of variables and evaluation method

For any inspection a Robot Evaluation is conducted based on the need for various stakeholders within the different phases of inspection. This time, a criterion is carefully defined as per
respective variables. For instance, the working environment is vital when evaluating the inspection robot and it can only be precisely determined when all relevant variables such as radius, velocity, and adhesion coefficient among others are taken into consideration. In this respect, the variables are analyzed and classified to form the criteria for evaluation.

Table 1. Second classification of the variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure in pipeline</td>
<td>$P$</td>
</tr>
<tr>
<td>Robot's velocity</td>
<td>$V$</td>
</tr>
<tr>
<td>Adhesion coefficient</td>
<td>$\mu$</td>
</tr>
<tr>
<td>Mass of the fluid</td>
<td>$m$</td>
</tr>
<tr>
<td>Temperature in the pipeline</td>
<td>$C$</td>
</tr>
<tr>
<td>Movement directions</td>
<td>$M$</td>
</tr>
<tr>
<td>Friction force</td>
<td>$F_f$</td>
</tr>
<tr>
<td>Stiffness of connection</td>
<td>$S$, $S_i$</td>
</tr>
<tr>
<td>Obstacles</td>
<td>$O_b$</td>
</tr>
<tr>
<td>Output torque</td>
<td>$T$</td>
</tr>
<tr>
<td>Weight of obstacles</td>
<td>$W$</td>
</tr>
<tr>
<td>Degrees of freedom</td>
<td>$B$</td>
</tr>
<tr>
<td>Sum of all forces</td>
<td>$\gamma F$</td>
</tr>
<tr>
<td>Thrust force</td>
<td>$F_t$</td>
</tr>
<tr>
<td>Sum of all supporting forces</td>
<td>$\gamma N$</td>
</tr>
<tr>
<td>Length of the pipeline</td>
<td>$L$</td>
</tr>
<tr>
<td>Density of the fluid inside the pipeline</td>
<td>$\rho_f$</td>
</tr>
</tbody>
</table>

Table 2 shows the Criteria set up for the assessment of robots; their respective abbreviations as well as the affecting variables sets.

Table 2. Criteria of assessment and affecting variables

<table>
<thead>
<tr>
<th>Standards of Assessment</th>
<th>Affecting Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Working environment specifications (W/E)</td>
<td>R, V, m, C, K, ?</td>
</tr>
<tr>
<td>Commercial Availability (C/A)</td>
<td>S, PS, Com</td>
</tr>
<tr>
<td>Autonomy of the robot (Aut.)</td>
<td>L, #B, S, Com,</td>
</tr>
<tr>
<td>Energy supply (Ene.)</td>
<td>PS, T, $I_g$, Com</td>
</tr>
<tr>
<td>Control mechanism (Con.)</td>
<td>L, Com, T</td>
</tr>
<tr>
<td>Production Costs (Cost)</td>
<td>PS, K, $R_R$, #B</td>
</tr>
<tr>
<td>Hydrodynamics (Dyn.)</td>
<td>V, P, $I_g$, $F_r$, $F_r$, $R_r$, $?_f$</td>
</tr>
<tr>
<td>Locomotion efficiency (Loc.)</td>
<td>P, O, Ob, $F_r$, $?_r$, DOF, $?_r$, $r_x$</td>
</tr>
<tr>
<td>Maneuverability (Man.)</td>
<td>O, T, M, $R_R$, DOF, $?_r$, B; $r_x$</td>
</tr>
<tr>
<td>Detection technology (D/T)</td>
<td>Ob, L, R, K</td>
</tr>
</tbody>
</table>

4. Conclusion

With regards to the variables of classification and evaluation, there is a need for oil industry to take into consideration all variables when developing any projects and such a situation presents a big gap in the development of in-pipe inspection robots. There also exist other variables while they have never been genuinely considered in robotics projects of in-pipe inspections and
development. As regards the variables set discussed above, not all of them are considered in the literature. Four variables are offered in this work, which are:

- **S** = safety and accessibility of pipeline
- **f** = density of fluid inside pipeline / flow rate
- **C** = temperature inside pipeline
- **M** = movement directions

Safety and accessibility of a robot ought to be carefully evaluated in the development projects as it defines most of the usage of inspection robots within the pipeline systems. Moreover, safety is a core value for oil companies, which can be implemented for the robots as well. This also affects the cost and time efficiency of the inspection, especially in cases when a robot is stuck or damaged during the inspection, it takes at least hours of work and efforts to rescue the robot, while causing more interruption of the pipeline process. If safety measures are overlooked the robot inspections will consequently fail as the result of tough regulations within the petroleum industry. On the other hand, the less accessibility of robot inspection is likely to stop half way.

The density inside a pipe is vital as it directly affects the pressure there and it translates into locomotion efficiency associated with any Robot. The movement directions define the ability of a robot to move in different directions when carrying out Pipelines inspections.

The variable assessment provides a researcher with a better evaluation criterion and a method often used within the Pipeline Industry compared to traditional methods of evaluation. The evaluation method employed in this case can be applied as a decision-making tool for the stakeholder to select the best robot for their work. It can be used for any type of robot: both those at the concept phase and the ones in current usage. In brief, in-pipe inspection robots cannot be ignored within the oil and petroleum industry, and there is a need for more research to come up with better technologies for a similar purpose.

**References**


FINITE SIZE AND TOPOLOGICAL EFFECT 
IN GAUGED FOUR-FERMION INTERACTION MODEL

TOMOHIRO INAGAKI\textsuperscript{1,2}

\textsuperscript{1}Information Media Center, Hiroshima University, Higashi-Hiroshima, 739-8521, Japan
\textsuperscript{2}Core of Research for the Energetic Universe, Hiroshima University, Higashi-Hiroshima, 739-8526, Japan
\textit{e-mail: inagaki@hiroshima-u.ac.jp}

1. Introduction

Many models of the particle physics are constructed based on symmetry. In the ground state of the system some symmetry is spontaneously broken. The chiral symmetry plays an important role for the mass generation of fermion fields. The symmetry must be broken for a massive fermion. In QCD the chiral symmetry is dynamically broken by the non-vanishing vacuum expectation value for a composite operator constructed by a quark and an anti-quark.

The symmetry property of the ground state depends on the environment. It is expected that the broken symmetry is restored in some critical environments, small size, high temperature, high density and strong curvature. We often employ a four-fermion interaction model as a low energy effective model of the strong interaction and investigate the symmetry behavior.

In this article we focus on the finite size and topological effect to the chiral symmetry breaking. In Sec. 2 a gauged four-fermion interaction model is introduced as a simple model which induces the chiral symmetry breaking. In Sec. 3 we evaluate the ground state by observing the extremum of the effective potential. Finally, we will give some concluding remarks.

2. Four-fermion interaction model on $R^{D-1} \otimes S^1$

We start from a gauged four-fermion interaction model defined by the Lagrangian density,

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \sum_{k=1}^{N} \bar{\psi}_k i \gamma^\mu D_\mu \psi_k + \frac{\lambda_0}{2N} \left( \sum_{k=1}^{N} \bar{\psi}_k \psi_k \right)^2, \quad (1)$$

where $D_\mu$ represents the covariant derivative, $\lambda_0$ is a coupling constant, $N$ is the number of fermion species. This Lagrangian is invariant under the discrete chiral transformation,

$$\psi \rightarrow \gamma^5 \psi. \quad (2)$$

This $Z_2$ chiral symmetry prevent the Lagrangian from having a mass term.

In order to investigate the finite size and the topological effect we consider the model on the background, $R^{D-1} \otimes S^1$, where $R^{D-1}$ is a $(D-1)$-dimensional flat Minkowski space time and $S^1$ is the one dimensional sphere. The boundary condition for the compact direction, $x^{D-1}$, can be determined by the spin structure of the fermion. Here we set the boundary condition,

$$\psi(x^0, \cdots, x^{D-2}, x^{D-1} + L) = e^{i\delta} \psi(x^0, \cdots, x^{D-2}, x^{D-1}), \quad (3)$$
where $L$ is the size of the compact direction. It should be noted that the theory is equivalent with the finite temperature field theory for the anti-periodic boundary condition, $\delta = \pi$, [3].

According to the auxiliary field method, the Lagrangian (1) is rewritten as

$$\mathcal{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{N} \sum_{k=1}^{N} \bar{\psi}_k (i \gamma^\mu D_\mu - \sigma) \psi_k - \frac{N}{\lambda_0} \sigma^2.$$  \hspace{1cm} (4)

If the auxiliary field, $\sigma$, develops a non-vanishing vacuum expectation value, the fermion mass term is dynamically generated and the $Z_2$ chiral symmetry is spontaneously broken.

3. Effective potential analysis

We usually evaluate the vacuum expectation value for the auxiliary field, $\sigma$, by observing the minimum of the effective potential. As a review, see for example [4]. In the leading order of the $1/N$ expansion the effective potential is given by

$$\frac{V(\sigma)}{N} = \frac{1}{2\lambda_0} \sigma^2 - \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{\infty} \int \frac{d^{D-1}k}{(2\pi)^{(D-1)/2}} \ln[(k^2 + \sigma^2)L^2] + O \left( \frac{1}{N} \right),$$  \hspace{1cm} (5)

with

$$d^{D-1}k \equiv dk_0 \cdots dk_{D-2}, \quad k^2 \equiv k_0^2 + \cdots + k_{D-2}^2 + \omega_n^2, \quad \omega_n = \frac{2n\pi + \delta}{L}.$$  \hspace{1cm} (6)

In this expression the time component of $k$ is transformed as $k_0 \to -i\omega_n$.

The effective potential (5) contains an ultraviolet divergence. The divergence can be regulated by the dimensional regularization, i.e. computing as an analytic function of the spacetime dimension, $D$. The divergence can be eliminated by renormalizing the coupling constant. We introduce the renormalized coupling, $\lambda$, and the renormalization scale $\mu$,

$$\frac{1}{\lambda} \equiv \left( \frac{1}{\lambda} - \frac{1}{\lambda_c} \right) \mu^{D-2} \quad \text{with} \quad \frac{1}{\lambda_c} \equiv \frac{1-D}{(2\pi)^{(D-2)/2}} \Gamma \left( 1 - \frac{D}{2} \right).$$  \hspace{1cm} (7)

The extremum of the potential satisfies the gap equation,

$$\frac{1}{\lambda} - \frac{1}{\lambda_c} - \frac{\sqrt{2}}{L\mu(2\pi)^{(D-1)/2}} \Gamma \left( 3 - \frac{D}{2} \right) \sum_{n=-\infty}^{\infty} \left( \frac{\omega_n^2 + \sigma^2}{\mu^2} \right)^{(D-3)/2} = 0.$$  \hspace{1cm} (8)

To calculate the gap equation we perform the integration over $k_0, \cdots, k_{D-2}$ in Eq. (5). An existence of a non-vanishing real solution of the gap equation is a necessary condition for the auxiliary field, $\sigma$, to develop a non-vanishing expectation value at the ground state. In Ref. [1] the solution of the gap equation is evaluated as a function of the phase $\delta$ in a curved spacetime.

The broken symmetry can be restored with decreasing the coupling constant through the second order phase transition. The critical value of the coupling, $\lambda_{cr}$, is given by the $\sigma \to 0$ limit of the gap equation,

$$\frac{1}{\lambda_{cr}} - \frac{1}{\lambda_c} - \frac{\sqrt{2}(2\pi)^{(D-5)/2}}{(L\mu)^{D-2}} \Gamma \left( 3 - \frac{D}{2} \right) \left[ \zeta \left( 3 - D, \frac{\delta}{2\pi} \right) + \zeta \left( 3 - D, 1 - \frac{\delta}{2\pi} \right) \right] = 0.$$  \hspace{1cm} (9)

In order to find a contribution of the boundary condition we transform the fermion as,

$$\psi'(x) \equiv e^{i\theta(x)} \psi(x), \quad \text{with} \quad \theta(x^0, \cdots, x^{D-2}, x^{D-1} + L) = \theta(x^0, \cdots, x^{D-2}, x^{D-1}) + \eta.$$  \hspace{1cm} (10)

The boundary condition for the transformed fermion is given by,

$$\psi'(x^0, \cdots, x^{D-2}, x^{D-1} + L) = e^{i(\delta + \eta)} \psi'(x^0, \cdots, x^{D-2}, x^{D-1}).$$  \hspace{1cm} (11)

The transformation (10) changes the kinetic term for the fermion. It can be cancelled by the gauge transformation. Thus, the Lagrangian is invariant under the transformation (10). It means that the phase $\delta$ should be fixed to minimize the effective potential.
To find the ground state we numerically evaluate the effective potential (5) with the phase $\delta$ varies. Here we discuss the result for a small coupling case, $\lambda < \lambda_{cr}(L, \delta)$. Since the gap equation (8) has no real solution for $\lambda < \lambda_{cr}(L, \delta)$, the extremum of the effective potential is found at $\sigma = 0$, only. In Fig.1 shows the effective potential at $\sigma = 0$ as a function of the phase $\delta$. The minimum is observed at $\delta = \pi$. For a non-vanishing $\delta$ the fermion has to depend on the coordinate of the compact direction, $x^{D-1}$, to satisfy the boundary condition (3). We cannot avoid to consider the inhomogeneous distribution of the fermion.

Figure 1. $\delta$-dependence of the effective potential at $\sigma = 0$ for $D = 2.0$ (bottom at $\delta = \pi$), 2.5, 3.0 and 3.5 (top at $\delta = \pi$).

4. Conclusion

The effective potential of the gauged four-fermion interaction model has been investigated on the background $R^{D-1} \otimes S^1$. If the coupling constant is smaller than the critical one, the minimum of the effective potential is found at the anti-periodic boundary condition, $\delta = \pi$.

In the effective potential analysis, we assume that the configuration of the auxiliary field, $\sigma$, is constant. In Ref. [2] it is pointed out that an inhomogeneous configuration of the auxiliary field can be realized at the ground state for a four-fermion interaction model in two dimensions. To evaluate the inhomogeneous configuration of the auxiliary field we need to extend the analysis for the gauged four-fermion interaction model in arbitrary dimensions. This will be discussed elsewhere.

Keywords: Quantum field theory, chiral symmetry, spontaneous symmetry breaking, finite size effect, topological effect

AMS Subject Classification: 70S10, 81Txx.

References

DEVELOPMENT OF THE COMBINED ALGORITHM FOR INCREASING THE MEASUREMENT ACCURACY

M.M. ISAYEV\textsuperscript{1}

\textsuperscript{1}Institute of Control Systems of ANAS, Baku, Azerbaijan
e-mail: mezahir@bk.ru

\textbf{Abstract.} The paper deals with the solution of the problems of developing the combined algorithm for increasing the measurement accuracy, being implemented on the base of simple additives and multiplicative tests enabling to determine the measured non-electrical quantities according to the measurement results of additional tests being used to identify nonlinear functions of data-measuring systems transformation.

\textbf{Keywords:} Test method, increasing measurement accuracy, identification, nonlinear transformation function.

\textbf{AMS Subject Classification:} 65Yxx.

\section{1. Introduction}

Currently, widely developed algorithmic methods for increasing the measurement accuracy are based on the use of computational technology, mathematical modeling, and the implementation of special algorithms for processing measurement results. In most cases the construction of DMS based on these methods makes it possible to provide high accuracy of measurement results (MR) using low-current initial measurement systems (MS). During the measurement, additional information can be obtained due to the structural, temporal, or structural-temporal redundancy of DMS [2].

It should be noted that, at present, in connection with the world-wide growth in the automation of technological processes in the production and carrying out complex scientific experiments, the creation of high-precision DMS of non-electric quantities is of great importance, as a result, the efficiency of the methods for reducing the correlated component of DMS non-electric quantities errors becomes particularly important. This is due to the fact that, the accuracy of MR obtained by means of them is mainly determined by the metrological characteristics of the sensors being operating systems under the most unfavorable working conditions.

The purpose of the work is to investigate a new variety of the test method (TM) for increasing the measurements accuracy of DMS with nonlinear transformation functions (TF), based on the use of test sets in test algorithms.

\section{2. The solution of the problem}

The essence of this method is reduced to obtaining additional information enabling to implement the algorithm for increasing the measurement accuracy in the functioning process of data-measuring systems (DMS).

It should be noted that, the development of the methods for reducing the correlated component of errors is of great importance for DMS non-electrical quantities [1–4].
Generally, the parameters of the real TF MT are nonstationary random time functions. In connection with these, the MT error will also be a non-stationary random time function.

For real DMS, it can be represented by the following sum [1, 3]:

\[ \Delta y(t) = \Delta y(t) + \Delta y(t), \]  

(1)

where \( \Delta y(t) \) — is a stationary, centered, ergodic; however, \( \Delta y(t) \) — is nonstationary random time functions.

The presentation of \( \Delta y(t) \)in the form (1) makes it possible to separate the measurement inaccuracy into two components depending on the particular spectrum:

- auto-correlated component \( \Delta y(t) \) of error \( \Delta y(t) \) including systematic, progressive and slowly changing relative to time T of random errors;

- the component \( \Delta y(t) \) of error \( \Delta y(t) \) with a short autocorrelation time is a non-auto-correlated error component including random uncorrelated errors such as “white noise” caused by the intrinsic noise of electronic and semiconductor cells.

Currently, along with SM method, the methods of inverse transformations (IT) are often used to reduce the correlated component of MR error of DMS. The SM methods make it possible to increase the accuracy of MR due to the introduction of accurate SM into the initial MS without determining the real values of TF MS parameters.

3. Test method to increase the measurement accuracy for the identification of highly nonlinear TF of DMS

Note that, in many practical cases, the mathematical model (MM) of TF of the initial MS can be represented by polynomial functions in the following form [1, 2, 4]:

\[ y(x) = \sum_{i=1}^{n} a_i x^{i-1}. \]  

(2)

Equations (2) are the system of linear equations with respect to the parameters and, according to Cramer’s law, the solution of this system is the basic test equation (BTE) of DMS testing.

Moreover, depending on the nonlinearity of TF of initial MS, the number of tests used in DMS tests and their qualitative relationship (the ratio of the number of additive to the number of multiplicative ones, and so on), this algorithm will have this or other complexity of implementation relative to the unknown value \( x \).

It is known [1–4] that, additional test measurements \( n \) of \( A_j(x) \), \( j = x, \ldots, n \) tests are provided in order to exclude the influence of TF parameters \( a_i \) instability on the measurement accuracy in DMS testing. In this case, the following system of \( n + 1 \) equations is formed by relatively unknown equation \((n + 1)\) of \( x \), \( a_1, \ldots, a_n \).

Therefore, one of the main criteria for the test set optimality used to identify the nonlinear TF of DMS is the minimum degree of BTE obtained by its implementation. However, the problem of using the tests of a particular type should also be related to the achievable accuracy forming their additive and multiplicative constants \( \theta \) and \( K \). This circumstance is caused by the fact that, the accuracy of MR is determined, first of all, by the accuracy of the tests being formed in the system, by using the DMS testing.

Theoretical investigations of test methods for increasing the measurement accuracy based on the implementation of simple additive and multiplicative test sets in DMS are described in detail in [1, 4].

Therefore, in practice, as a rule, for a given TF of initial MS source tends to somehow reduce the degree of BTE relative to the measured quantity and, thereby, narrow the domain of the desired root. For this, TF of DMS over the entire range is divided into nonlinear sections, between the maximum and minimum points.
In many cases, practical implementation of DMS non-electrical quantities of latter TF can be described with sufficient accuracy by piecewise polynomials of the second degree in the following form:

$$y = a_1s + a_2sx + a_3sx^2,$$

where $s = 1, \ldots, \ell^*$ is the area of approximation.

Based on this, the synthesis of the test algorithm in this DMS is implemented as follows. For $n = 3$ the optimal test $(x + \theta; kx; kx + \theta)$ set is determined by [1, 3]. Based on it, the structure of DMS testing (DMST), enabling to implement one additive, one multiplicative and one combined test in the system, is determined. Then, in addition to known $n = 3$ and definite test set, there are basic test equations (BTE) for this DMST:

$$y_0 = \frac{x(k - 1) + \theta}{x(k - 1) - \theta} (y_1 - y_2) + y_3(xk - x - \theta).$$

In addition, the algorithm for processing the measurement results (MR) of the tests used in the system with respect to the value $x$ is determined as follows:

$$x_{\text{comp}} = \frac{(y_1 - y_2) + (y_0 - y_3)}{(y_0 - y_3) - (y_1 - y_2) (k - 1)} \theta. \quad (3)$$

The measurement process consists of four cycles. In the first cycle the investigated parameter $x$ is being measured, in the second cycle the additive test $x + \theta$, in the third cycle the multiplicative test $xk$, and in the fourth cycle the combined test $xk + \theta$ is being measured.

The relations combining MR with measured quantities $x + \theta, xk$ and $xk + \theta$ have the following form:

$$\begin{align*}
y_0 &= a_1s + a_2sx + a_3sx^2 \\
y_1 &= a_1s + a_2s(x + a_3s + \theta) + (x + \theta)^2 \\
y_2 &= a_1s + a_2sxk + (xk)^2 \\
y_3 &= a_1s + a_2s(xk + \theta) + a_3s(xk + \theta)^2.
\end{align*}$$

The ECM of DMS, processing the obtained measurement results $y_0, \ldots, y_3$ by the algorithm (3), determine the measured quantity $x$.

When the nonlinearity of TF of initial MS is necessary to use polynomials of the third, fourth, and so on for the crochet approximation, BTE DMST becomes nonlinear in accordance with the measured quantity and its solution can be realized by iterative algorithms [2] or numerical methods on ECM.

4. Conclusion

Thus, as it can be seen from the foregoing, the application the developed methods of constructing algorithms for the operation MS allows to obtain a number of new qualities in a new modification of test systems along with the merits that are characteristic of the DMS, implemented on the base of simple additive and multiplicative tests.

References

SUMMATION THE STATISTICAL INDICATORS AND FORMING THE DATABASE FOR WELLS COVERING THE SAME LAYER

N.A. ISMAILOV\textsuperscript{12}, F.A. ALIEV\textsuperscript{12}, I.A. MAHARRAMOV\textsuperscript{1}, K.G. GASIMOVA\textsuperscript{1}

\textsuperscript{1}Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
\textsuperscript{2}Institute of Information Technology of ANAS, Baku, Azerbaijan
\textsuperscript{3}Azerbaijan State Pedagogical University, Baku, Azerbaijan
e-mail: f_aliev@yahoo.com

Usually the creation of a model, describing this or another object, is based on the input-output parameters collected during observations of this object (or to the created database) \cite{2, 3, 4}. The creation of a mathematical model is carried out by two methods (or a combination of these methods).

In the first method, the system is divided into such subsystems that their properties and features are known from earlier conducted or collected experiments and observations. Combination formally of these subsystems from a mathematical point of view, in the end leads to the formation of a mathematical model of the entire system. In the framework of such approach, in the construction of a mathematical model of an object, experiments are not necessary, i.e. in this case everything corresponds to the "laws of nature". In this case, the procedures of mathematical modeling depend on a specific applied mathematical problem and are usually determined by the customary specific features of the considered applied area.

Another method of constructing a mathematical model is based on materials extracted or collected from experiments. In this case, using the database processing, built on the basis of the results of the input-output signals, a model is formed. And this is the method of identification. Identification of the system, i.e. The construction of the model from the observations is composed of three main components \cite{3, 4}.

- Data;
- Models describing the process;
- Evaluation or selection of an approved model on the base of experiments.

At the first stage, the statistical wells data were processed and compared with the results of the model and carried out the following works. The development of information (observations or statistical indicators) plays an important role both in the practice in applying of mathematical calculations and in financial and economic calculations. The development of signals and observations is realized with the help of some algorithms in the MATLAB package \cite{1, 2}. All of the above was carried out on the basis of statistical indicators of well No. 248 (suspended) of DPOG (NGDU) Narimanova SOCAR for the years 1976-2012.

Above these statistics, some groupings were carried out and they are listed in the table form and are reflected in the MATLAB package in the following form:

\[
\begin{array}{cccc}
\end{array}
\]
The graphs reflecting the dynamics of the change in the amount of gas consumed and the resulting oil are obtained.

Visual examination of these graphs shows that during the observation of statistical indicators the numerous noise were admitted. So, a database was created that reflects statistical indicators and was constructed the dynamics of the well production for the period corresponding to this base.

**Keywords:** Gas lift, gas consumption.

**AMS Subject Classification:** 49J15, 49J35.

**References**

ON EXTREMLITY OF SOME ALGEBRAIC VARIETIES

I. SH. JABBAROV¹, G. K. HASANOVA¹

¹Ganja State University, Ganja, Azerbaijan
e-mail: ilgar_j@rambler.ru

1. Introduction

In the work (see [5]) Khintchine A. had considered the metric questions of the theory of Diophantine Approximation of dependent quantities. He showed that the system of inequalities
\[ \max(\|tq\|, \|t^2q\|, \ldots, \|t^nq\|) < \delta q^{-1/n} \]
has infinite set of solutions in positive integral numbers \( q > 0 \) for almost all real \( t \) in the Lebesgue sense. To define basic notions of the theory, we consider the system of inequalities
\[ \max(\|\alpha_1 q\|, \|\alpha_2 q\|, \ldots, \|\alpha_n q\|) < q^{-u}, \ u > 0, \]  
\[ (1) \]
where \( \|\alpha\| \) means a distance from \( \alpha \) to the nearest integral number.

Let \( u(\alpha_1, \ldots, \alpha_n) \) be defined as a sup of such \( u > 0 \) for which (1) is satisfied for infinite set of natural numbers \( q \). The inequality \( u(\alpha_1, \ldots, \alpha_n) \geq 1/n \) is known (see [6]). Then we can state that the inequality (1) is satisfied for infinitely many natural numbers \( q \) when \( u < 1/n \). When \( u(\alpha_1, \ldots, \alpha_n) = 1/n \) for almost all points of the variety \( (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \) of less dimension, then we call it an extremal variety.

There is a close connection between the system of inequalities (1) and an inequality
\[ |\alpha_1 a_1 + \cdots + \alpha_n a_n| < a^{-v}, \]
which assumed to have infinite number of solutions in the system of integral numbers \( (a_1, \ldots, a_n) \), for given \( v > 0 \). As in the case of the system of inequalities above, we define the number
\[ v(\alpha_1, \ldots, \alpha_n) = \sup w, \]
where the sup is taken over all such positive \( w \) for which the inequality above has infinite set of solutions. From the Dirichlet’s principle it follows that \( v(\alpha_1, \ldots, \alpha_n) \geq n \). As it is best known (see [4]), there is following relation between two numbers \( u(\alpha_1, \ldots, \alpha_n) \) and \( v(\alpha_1, \ldots, \alpha_n) \), namely the relation \( v(\alpha_1, \ldots, \alpha_n) = n \) is equivalent to the inequality \( u(\alpha_1, \ldots, \alpha_n) = 1/n \). This relation lets us to establish a close connection between the theory on extremality of algebraic varieties and the metric theory of transcendental numbers.

First example of one dimensional extremal variety in \( \mathbb{R}^2 \) with \( \alpha_1 = t, \alpha_2 = t^2 \) was given by Kubiluce J. P. (see [6]). Later in [2, 6] were got another examples of extreme varieties. Most important result was got by Sprindzuk V. G. by using the method developed by him and called the method of substantive and non-substantive domains.

In 1993 Karatsuba A. A. has supposed that the question on extremality of some algebraic varieties could investigated by using of results on convergence exponent in the Tarrys problem. In the present work we show that this proposition is valid and we show that the most of algebraic
varieties defined by the theorem below are extremal. This result generalizes many known results. So, we get the result of Sprindzuk V. G. by a new method.

2. Auxiliary results

Following result is a variant of the Borel-Kantelly lemma and plays important role in the questions concerning extremality of algebraic varieties (see[6]).

**Lemma 1.** Let $A_q (q = 1, 2, \ldots)$ be a sequence of measurable sets in $\mathbb{R}^n$, and

$$\sum_{q=1}^{\infty} (\text{mes} A_q)^m < \infty,$$

for some natural $m$. Then the measure of such real numbers which fall into infinite number of sets $A_q$ equals to zero.

**Proof.** Let’s designate $E \subset \mathbb{R}^n$ the subset which is a subset of infinitely many sets $A_q$. Suppose, in contrary, that $\text{mes} E \neq 0$. Then there exist a measurable subset $E_0 \subset E$ such that $\text{mes} E_0 = \mu > 0$. Let the sets containing the subset $E_0$ be $A_{q_1}, A_{q_2}, \ldots$. Then, for every $q_s$ we have

$$\sum_{s=1}^{\infty} \mu^m < \sum_{q=1}^{\infty} (\text{mes} A_q)^m < \infty.$$  

This is a contradiction. Lemma 1 is proven.

Note that the variant we have proven above is more useful for applications. In many questions it is possible more accurately estimate some positive power of the interesting measure. The series of measures stands convergent after of taking great power.

Our second auxiliary tool is an integral analog of Kovalevskaya lemma.

**Lemma 2.** Let $m, n, q$ be natural numbers, $f_j (\bar{x}), j = 1, \ldots, N$ be a real measurable functions defined in the cube $\Omega = [0, 1]^r$, $1 \leq r \leq N$. Denote by $\mu(q)$ the measure of a set of that $\bar{x} \in \Omega = [0, 1]^r$ for which

$$\max_j \| f_j (\bar{x}) \| < q^{-r_j} (1 \leq j \leq N).$$

Then,

$$\mu(q) << q^{-r} \times$$

$$\int_{(2\pi N)^{-1} q^r}^{(2\pi N)^{-1} q^r} \cdots \int_{(2\pi N)^{-1} q^r}^{(2\pi N)^{-1} q^r} \int_E e^{2\pi i (\alpha_1 f_1 (\bar{x}) + \cdots + \alpha_N f_N (\bar{x}))} d\bar{x} \, d\alpha_1 \cdots d\alpha_N;$$

here $r = r_1 + \cdots + r_N$, and the constant in the symbol $<<$ depends on $N$ only.

Our next result is a result on the convergence exponent of the special integral of Tarry’s problem (see [3-6]). The number $\gamma$ is called to be the convergence exponent for the integral

$$\theta_0 = \theta_0 (k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^1 e^{2\pi i (\alpha_1 x_n + \cdots + \alpha_1 x_1)} dx \, d\alpha_n \cdots d\alpha_1$$

if it converges when $2k > \gamma$ and diverges when $2k < \gamma$.

The lemma 2 allows us to reduce the estimation of measures from the lemma 1 to the question on convergence exponent for the special integral.

**Lemma 3.** We have $2k > 1 + n(n + 1)/2$ for the integral (2). Proof of this lemma is given in [1].
3. Basic result

**Theorem.** The algebraic variety \( \Gamma = (x, x^2, \ldots, x^n) \) is extremal. **Proof.** We take in the lemma \( 2 \; r_j = 1/N + \delta, \; \delta > 0, \) and \( f_j(x) = qx^j \; j = 1, \ldots, N. \) If \( q \) is a natural number then for the measure \( \mu(q) \) of a set of that \( x \in [0,1] \) for which

\[
\max_j \| qx^j \| < q^{-r_j},
\]

we must have

\[
\mu(q) << q^{-(1-N-N\delta)K} \times \int_{-q^{1+1/N+\delta}}^{q^{1+1/N+\delta}} \cdots \int_{-q^{1+1/N+\delta}}^{q^{1+1/N+\delta}} \left| \int_{\Omega} e^{2\pi i (qa_1x + \cdots + qa_Nx^N)} \, dx \right| \, d\alpha_1 \cdots d\alpha_N,
\]

and the constant in the symbol \( << \) depends on \( N \) only. Applying the Holder’s inequality we find:

\[
\mu(q)^k << q^{-(kN - kN\delta)2^{N}K(1+N+N\delta)(K-1)} \times \left( \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \int_{\Omega} e^{2\pi i (qa_1x + \cdots + qa_Nx^N)} \, dx \right|^k \, d\alpha_1 \cdots d\alpha_N \right).
\]

As it was shown in [3] When \( k > \gamma \) the inner integral is convergent. So, we get

\[
\mu(q)^k << q^{-(1-N-N\delta)}.
\]

Applying the lemma 1 with \( m = k \), we conclude that the series of this lemma is convergent. Then the set \( E \subset \Omega \) for the points of which the conditions

\[
\max_j \| \gamma_j(x) \| < q^{-1/N-\delta} \quad (1 \leq j \leq n)
\]

Proof of the theorem 2 is completed.

4. Conclusion

In the paper it is considered the question on extremality of algebraic variety. We show validity of Karatsuba’s conjecture on possibility for the proof of extremality using of results on convergence exponent of special integral of Tarry’s problem.

**Keywords:** Diophantine approximation, algebraic variety, extremal variety, convergence exponent.

**AMS Subject Classification:** 11J17, 11J83, 11K60.

**References**


INVESTIGATION AND SOLVING SOME FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS BY SPECTRAL AND CONTOUR INTEGRAL METHODS

F. JAHANSHahi1, M. JAHANSHahi

1Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
e-mail: f.jahanshahi@azaruniv.ac.ir and jahanshahi@azaruniv.edu

ABSTRACT. In this paper, we consider an initial - boundary value problem which contains a fractional differential equation by some initial and boundary conditions by spectral method and contour integral method.

Keywords: Initial-boundary value problem, laplace transformation, fractional equation, contour integral.

AMS Subject Classification: 53C25, 53C40.

1. INTRODUCTION

At first, the given fractional equation is without mixed term derivative. We get its spectral problem and the eigenvalues and eigenfunctions. In the second part, we consider this equation with a mixed term derivative of time variable and space variable. In this case, we can not use the spectral method. Its spectral problem is obtained by Laplace transformation. The analytic solution of this problem is written as integral form over a suitable contour that its Green function is the solution of related spectral problem.

2. THE FRACTIONAL EQUATION WITHOUT MIXED TERM DERIVATIVE

In first case, the fractional equation with initial and boundary conditions, has no mixed term derivative as follows:

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial x} + cu ,
\]

\[
u(0, t) = 0, \quad u(1, t) = 0,
\]

\[
u(x, 0) = \varphi(x),
\]

where \(\alpha\) is the fractional order derivative. For this equation, without lose of generality, if we suppose \(a = 1, b = 2a, c = 0\) in the equation(1), the spectral problem of (1) will be in the following form:

\[
y''(x) + 2ay'(x) - \lambda^2 y(x) = 0, \quad x \in (0, 1),
\]

\[
y(0) = 0, \quad y(1) = 0,
\]
the solution of related spectral problem is:
\[ y(x) = c_1 e^{(-a-\sqrt{a^2+k^2\pi^2})x} + c_2 e^{(-a+\sqrt{a^2+k^2\pi^2})x}, \]
its eigenvalues and eigenfunctions are determined by:
\[ \lambda_k = \sqrt{-(a^2+k^2\pi^2)}, \ k \in \mathbb{Z}, \]
\[ y_k(x) = c_1 e^{-ax} \sin k\pi x, \ k \in \mathbb{N}. \]
For adjoint problem and its eigenvalues and eigenfunctions, we have:
\[ Z'' - 2aZ' - \rho^2 Z = 0, \ Z(0) = 0, \ Z(1) = 0, \]
\[ \rho_k = \sqrt{-(a^2+k^2\pi^2)}, \ k \in \mathbb{N}, \ Z_k(x, \rho) = e^{ax} \sin k\pi x. \]
Then we construct its approximate solution in the following form:
\[ u(x, t) = \sum_{k=1}^{\infty} T_k h_k(t, \lambda_k) y_k(x, \lambda_k), \]
where \( h_k(t, \lambda_k) \) are Mittag Leffler functions. By imposing the initial condition in (10), we have:
\[ u(x, 0) = \varphi(x) = \sum_{k=1}^{\infty} T_k h_k(0, \lambda_k) y_k(x, \lambda_k). \]
To sum up, we conclude following theorem:

**Theorem 1.** For the Initial-boundary value problem (1)-(3) and its spectral problem (4)-(5) with adjoint problem (9), we have its approximate solution in from of (10), where
\[ T_k = \langle \varphi, Z_m(x, \rho_m) \rangle. \]

### 3. The Fractional Equation with Mixed Term Derivative

In this section, we consider the fractional PDE with mixed term derivative as follows:
\[ D^2 u_{xx}(x, t) + aD^{1+\alpha} u_{xt}(x, t) + bD^\alpha u_x(x, t) + cD^\alpha u_t(x, t) + du(x, t) = 0, \ x \in (0, 1), \ t > 0, \]
\[ u(0, t) = 0, \ u(1, t) = 0. \]
Since we can not apply the fourier method, for this case, the spectral problem is obtained by Laplace transformation method as follows:
\[ y''(x, \lambda) + (a\lambda^\alpha + b)y'(x, \lambda) + (c\lambda^\alpha + d)y(x) = f(x), \ x \in (0, 1), \]
\[ y(0) = 0, \ y(1) = 0, \]
where its characteristic equation is:
\[ \theta^2 + (a\lambda^\alpha + b)\theta + (c\lambda^\alpha + d) = 0, \]
assuming \( \lambda^\alpha = \rho \), we have:
\[ \theta(\lambda^\alpha) = \theta(\rho) = \frac{-(a\rho + b) \pm \sqrt{(a\rho + b)^2 - 4(c\rho + d)}}{2}, \]
\[ y(x, \lambda) = c_1 e^{\theta(\rho)x} + c_2 e^{\theta(\rho)x}. \]
Their asymptotic of these roots are:
\[ \theta_1(\lambda^\alpha) = -a\lambda^\alpha - \frac{ab - c}{a} - \frac{abc - c^2 - da^2}{a^3\lambda^\alpha} - \frac{A_{-2}}{\lambda^{2\alpha}} + \ldots, \]
\[ \theta_2(\lambda^\alpha) = -\frac{c}{a} + \frac{abc - c^2 - da^2}{a^3\lambda^\alpha} + \frac{A_{-2}}{\lambda^{2\alpha}} + \ldots. \]
The solution of the spectral problem is written analytically by integral method as follows:
\[ y(x, \lambda) = c_1 e^{\theta_1 x} + c_2 e^{\theta_2 x} + \int_0^1 g(x, \xi, \lambda^\alpha) f(\xi) d\xi , \] (18)
by imposing boundary conditions (15) in (18), its eigenvalues are determined by:
\[ \lambda_k^\alpha = \frac{2c}{a^2} - \frac{b}{a} \pm \frac{2k\pi i}{a} . \] (19)
Finally, the solution of main problem (12)-(13) is given by contour integral method as follows:
\[ u(x, t) = \frac{1}{2\pi i} \int_L e^{\lambda t} y(x, \lambda^\alpha) d\lambda , \] (20)
where
\[ y(x, \lambda^\alpha) = \int_0^1 G(x, \xi, \lambda^\alpha) f(\xi) d\xi . \]

**Theorem 2.** For the Initial-boundary value problem (12)-(13) and its spectral problem (14)-(15), with the eigenvalues (19), we have: a) If the real part of \( \lambda^\alpha \) is zero, that is:
\[ \frac{2c}{a^2} - \frac{b}{a} = 0 \Rightarrow ab = 2c , \] (21)
then the eigenvalues are in imaginary axis, in other words the all of eigenvalues lie in left-hand side of Laplace line \( x = c \) and therefore the analytic solution can be written by Laplace inverse transformation as follows:
\[ u(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} y(x, \lambda^\alpha) d\lambda . \] (22)
b) If the real part of eigenvalues \( \lambda^\alpha \) is not zero, that is:
\[ ab \neq 2c \]
then all of eigenvalues lie in whole of complex plane and the analytic solution can be written by following contour integral form:
\[ u(x, t) = \frac{1}{2\pi i} \int_L e^{\lambda t} y(x, \lambda^\alpha) d\lambda , \] (23)
where \( L \) is a closed contour line.

4. Conclusion

In this paper, we discussed some Initial-boundary value problems including time fractional PDEs. At first their spectral problem were obtained. According to the asymptotic expansions of eigenvalues of their spectral problems, their analytic solutions were presented as Laplace inverse transformation and closed contour integral line.

**References**

SPECTRAL PROBLEM FOR AN INITIAL-BOUNDARY VALUE PROBLEM INVOLVING FIRST ORDER TWO DIMENSIONAL GENERALIZED NONHOMOGENOUS CAUCHY-RIEMANN EQUATION WITH GENERAL NON-LOCAL BOUNDARY CONDITIONS

M. JAHANSHAHI\textsuperscript{1}, J. EBAZPOUR GOLANBAR\textsuperscript{2}, N. ALIYEV\textsuperscript{3}

\textsuperscript{1}Dept of Mathematics- AzarbaijanShahidMadani University, Tabriz, Iran
\textsuperscript{2}Dept of Mathematics- AzarbaijanShahidMadani University, Tabriz, Iran
\textsuperscript{3}Dept of Mathematic, Baku State University, Baku, Azerbaijan
e-mail: jahanshahi@azaruniv.edu

Abstract. In this paper, we consider an initial boundary value problem which contains a fractional differential equation by some initial and boundary conditions by spectral method. This problem is considered in two cases. At first, the given fractional equation is whit out mixed term derivative. We get its spectral problem and the eigenvalues and eigenfunctions. In the second part, we consider this equation whit a mixed term derivative of time variable and space variable. In this case, we cannot use the spectral method. Its spectral problem is obtained by Laplace transformation. The analytic solution of this problem is written as integral form over a suitable countour that its Green function is the solution of related spectral problem.

Keywords: Boundary value problem, generalized Cauchy-Riemann equation, compatibility conditions, Fredholm integral equation, regularization, weak singularities.

AMS Subject Classification: 35E05, 35J56, 45E99.

1. Introduction

We will make the spectral problem for the given initial-boundary value by Laplace transformation. The resulted spectral problem will be in form of boundary value problem which consist of generalized Cauchy-Riemann equations with parameter \( \lambda \).

This problem and BVP are reduced to system of second kind regularized Fredholm integral equations by a special technique which is related to authors \cite{4, 5}. Therefore the main goal of this paper is construction of related Spectral problem. At first we provide the adjoint problem of Spectral and then we calculate the fundamental solution of adjoint problem. Finally, we identity the singularities in integrals and obtaining necessary conditions which are used to reduce to the second kind regularized integral equations as same as in \cite{1}.
2. Mathematical statemental problem and its spectral problem

In this paper, we consider the following initial and boundary value problem:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x_2} + i \frac{\partial u(x,t)}{\partial x_1} + a(x)u(x,t) + f(x,t) \quad x \in D \subseteq \mathbb{R}^2, t > 0$$

with boundary and initial conditions:

$$\alpha_1(x_1) u(x_1, \gamma_1(x_1), t) + \alpha_2(x_1) u(x_1, \gamma_2(x_1), t) = \alpha(x_1, t) \quad x_1 \in [a_1, b_1], t > 0,$$

$$u(x,0) = \varphi(x) \quad x \in \overline{D}$$

we obtain its spectral problem by Laplace transformation as follows:

$$\frac{\partial \tilde{u}(x,\lambda)}{\partial x_2} + i \frac{\partial \tilde{u}(x,\lambda)}{\partial x_1} - \lambda \tilde{u}(x,\lambda) = F(x,\lambda),$$

$$\alpha_1(x) \tilde{u}(x_1, \lambda_1(x_1), \lambda) + \alpha_2(x) \tilde{u}(x_1, \gamma_2(x_1), \lambda) = \tilde{\alpha}(x_1, \lambda).$$

By using operator representation, we write

$$\ell \tilde{u} \equiv \frac{\partial \tilde{u}}{\partial x_2} + i \frac{\partial \tilde{u}}{\partial x_1} - \lambda \tilde{u}$$

and also,

$$\ell \tilde{u} \equiv F(x,\lambda).$$

3. Adjoint problem for Spectral problem and its fundamental solution

Suppose $V$ is a complex arbitary function, and $\overline{V}$ is complex conjugate of $V$. Then by multiplying $\overline{V}$ to the both sides of differential equation (6) and integration over region $D$, we have:

$$(\ell \tilde{u}, v) = \int_D \frac{\partial \tilde{u}}{\partial x_2} V(x,\lambda)dx + i \int_D \frac{\partial \tilde{u}}{\partial x_1} V(x,\lambda)dx - \lambda \int_D \tilde{u}(x,\lambda) V(x,\lambda)dx.$$

Therefore by considering the above relation and some calculation by using Gauss-Stragradesky formula, we obtain the related adjoint equation:

$$\ell^* V \equiv - \frac{\partial V(x,\lambda)}{\partial x_2} + i \frac{\partial V(x,\lambda)}{\partial x_1} - \overline{V}(x,\lambda)$$

and its fundamental solution in the following form:

$$V(x - \xi, \lambda) = \frac{-1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{i(x,\lambda) - \xi}}{i\alpha_2 + \alpha_1 + \lambda} d\alpha,$$

that is:

$$\ell^* V(x - \xi, \lambda) = \delta(x - \xi),$$

where $\ell^*$ is the adjoint equation (4) and $\delta(x - \xi)$ is the Delta-Dirac function.
4. Compatibility conditions

As applied in [4, 5, 7] for obtaining the compatibility conditions (or necessary conditions), the fundamental solution \( V \) is multiplied to the both sides of equation (4), then is integrated over \( D \), that is:

\[
\int_D \frac{\partial \tilde{u}(x, \lambda)}{\partial x_2} V(x - \xi, \lambda) dx + i \int_D \frac{\partial \tilde{u}(x, \lambda)}{\partial x_1} V(x - \xi, \lambda) dx \\
- \lambda \int_D \tilde{u}(x, \lambda) V(x - \xi, \lambda) dx = \int_D F(x, \lambda) V(x - \xi, \lambda) dx.
\]

By using Gauss-Astragradeski formula, we have:

\[
\frac{1}{2} \tilde{u}(\xi_1, \gamma_k(\xi_1), \lambda) = \int_D F(x, \lambda) V(x_1 - \xi_1, x_2 - \gamma_k(\xi_1), \lambda) dx \\
- \int_{a_1}^{b_1} \tilde{u}(x_1, \gamma_1(x_1), \lambda) V(x_1 - \xi_1, \gamma_1(x_1) - \gamma_k(\xi_1), \lambda) [i \gamma_1'(x_1) - 1] dx_1 \\
- \int_{a_1}^{b_1} \tilde{u}(x_1, \gamma_2(x_1), \lambda) V(x_1 - \xi_1, \gamma_2(x_1) - \gamma_k(\xi_1), \lambda) [1 - i \gamma_2'(x_1)] dx_1,
\]

\( \xi_1 \in (a_1, b_1), k = 1, 2 \).

By applying these compatibility conditions the spectral problem (4), (5) is converted to a second kind Fredholm integral equations with weak singularities according to the following results:

5. Main results

**Theorem 1.** For the Initial-Boundary value problem (1) – (3), suppose the region \( D \) is a planar and bounded subset in \( \mathbb{R}^2 \) with Lyapunov boundary line

\( \Gamma = \mathcal{D} \setminus D = \Gamma_1 \cup \Gamma_2 \),

if the functions \( \alpha(x) \), \( f(x, t) \) are known and continuous functions on \( \mathcal{D} \), \( \alpha_1(x_1, t) \) and \( \alpha_p(x_1) \), \( p = 1, 2 \) are Holder functions. Then every solution of equation (4) satisfies in compatibility condition (5).

**Theorem 2.** Under hypothesis of theorem 1,

a) The Spectral problem (4) - (5) of main initial-boundary value problem (1) – (3) reduces to the second kind regularized integral boundary equations.

b) The solution of main problem (1) – (3) will be given by following expressions:

\[
\tilde{u}(\xi, \lambda) = g(\xi, \lambda) + \int_D R(\xi, \theta, \lambda) g(\theta, \lambda) d\theta,
\]

where \( g(\xi, \lambda) \) the Fourier transformation of \( g(x) \) and

\[
u(\xi, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} \tilde{u}(\xi, \lambda) d\lambda,
\]

(11)
or

\[
u(\xi, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} [g(\xi, \lambda) + \int_D R(\xi, \theta, \lambda) g(\theta, \lambda) d\theta] d\lambda,
\]

(12)

where the kernel \( R(\xi, \theta, \lambda) \) has no singularity.
References


APPROXIMATE SOLUTIONS FOR SOME NONLINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS BY ADOMIAN DECOMPOSITION METHOD AND MITTAGE-LEFFLER FUNCTIONS

M. JAHANSHAHI, H. KAZEMI DEMNEH

1Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
 e-mail: jahanshahi@azaruniv.edu, kazemi.hamdam@azaruniv.ac.ir

Abstract. In this paper, a class of nonlinear fractional partial differential equation is considered and solved by advanced analytical-numerical methods such as Adomian Decomposition Methods and Mittag-Leffler functions. The obtained approximate solutions show that these solutions are same for the first three approximate terms $u_1, u_2, u_3$.

Keywords: Nonlinear fractional differential equation, Mittag-Leffler functions, adomian decomposition method.

AMS Subject Classification: 35M33.

1. Introduction

Various analytical methods, for example, Laplace and Fourier transforms, have been utilized to solve linear fractional differential equations [1,3], but for solving nonlinear fractional differential equations, numerical methods have been used solely. In this paper, we solve the following nonlinear fractional partial differential equation with Adomian Decomposition Methods and Mittage-Leffler functions:

\[ D_t^\alpha u(x, t) = u(x, t) + u^n(x, t). \]  

(1)

2. Solving problem with Adomian Decomposition Method

Consider the following nonlinear fractional differential equation

\[ D_t^\alpha u(x, t) = u(x, t) + u^2(x, t). \]  

(2)

First, we convert the equation(2) to a fractional integral equation, then we solve the integral equation with Adomian Decomposition Method (ADM). Now, By integrating both sides of the equation (2), the order of $\alpha - 1$ ( with respect to time variable $t$) we obtain:

\[ D_t^{1-\alpha} D_t^\alpha u(x, t) = D^{1-\alpha} u(x, t) + D_t^{1-\alpha} u^2(x, t), \]  

(3)

so we have:

\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} u(x, \tau) d\tau + \frac{\partial}{\partial t} \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} u^2(x, \tau) d\tau, \]  

(4)

then, by integrating both sides of(4) the in interval $[0,t]$, we have:

\[ u(x, t) = u(x, 0) + \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} u(x, \tau) d\tau + \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} u^2(x, \tau) d\tau. \]  

(5)
Now, we solve the above integral equation with the initial condition \( u(x, 0) = \varphi(x) \). Due to the nonlinear term \( u^2 \), according to the general ADM, we have:

\[
\sum_{n=0}^{\infty} u_n(x, t) = \varphi(x) + \sum_{n=0}^{\infty} \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} [u_n(x, \tau) + A_n(x, \tau)] d\tau.
\] (6)

In this case, we have:

\[
u_{n+1}(x, t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{(\alpha-1)!} [u_n(x, \tau) + A_n(x, \tau)] d\tau.
\] (7)

Due to the fractional derivative and integral definitions [3], we have:

\[
u_{n+1}(x, t) = I^\alpha(u_n(x, t) + A_n(x, t)),
\] (8)

so

\[
u_1(x, t) = I^\alpha(u_0(x, t) + A_0(x, t)) = I^0_0((\varphi(x) + \varphi^2(x)) = I^0_0(\varphi + \varphi^2)
\]

\[
u_2(x, t) = I^\alpha(u_1(x, t) + A_1(x, t)) = I^0_0(I^0_0(\varphi + \varphi^2) + 2\varphi I^0_0(\varphi + \varphi^2)).
\]

Due to the following relation:

\[
D^\alpha_0(a D^\frac{q}{p} f(t)) = a D^\frac{q}{p} f(t),
\] (9)

then \( u_2(x, t) \) be obtained as follows:

\[
u_2(x, t) = I^0_0(\varphi + \varphi^2) + 2\varphi I^0_0(\varphi + \varphi^2) = (1 + 2\varphi) I^2_0(\varphi + \varphi^2).
\] (10)

Similarly

\[
u_3(x, t) = I^0_0(u_2(x, t) + A_2(x, t)) = I^0_0((1 + 2\varphi) I^2_0(\varphi + \varphi^2) + 2\varphi(1 + 2\varphi) I^2_0(\varphi + \varphi^2) + (I^0_0(\varphi + \varphi^2))^2),
\] (11)

and as a result

\[
u_3(x, t) = (1 + 2\varphi)^2 I^3_0(\varphi + \varphi^2) + I^0_0(\varphi + \varphi^2)(I^0_0(\varphi + \varphi^2))^2.
\] (12)

Consider the problem (2) with the initial condition \( u(x, 0) = \varphi(x) \). We see that solving problem with ADM leads to obtain the fractional integrals \( I^0_0, I^2_0, I^3_0 \). So, we obtain them as follows.

We calculate \( I^1_0(\varphi + \varphi^2), I^3_0(\varphi + \varphi^2), I^3_0(\varphi + \varphi^2) \) then by putting these sentences into \( u_1(x, t), u_2(x, t), \ldots \), the approximate solution \( u(x, t) \) is obtained.

**Example 1.** Let \( \alpha = \frac{1}{2}, \varphi(x) = x \), by making use of the following relation

\[
I^\alpha x^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \alpha + 1)} t^{\lambda + \alpha}, \quad \lambda > -1, \alpha > 0,
\] (13)

and considering to the properties of the Gamma function, we have:

\[
I^\frac{1}{2} (x + x^2) = \frac{2(x^2 + x^2 t^{\frac{3}{2}})}{\sqrt{\pi}},
\]

\[
I^\frac{3}{2} (x + x^2) = I^\frac{1}{2} (I^\frac{3}{2} (I^\frac{1}{2} (\frac{2(x^2 + x^2 t^{\frac{3}{2}})}{\sqrt{\pi}}))) = \frac{2(x + x^2)}{\sqrt{\pi}} I^\frac{1}{2} (t^{\frac{3}{2}})
\]

Now, by putting the resulted fractional derivative (8), (12), we have:

\[
u_1(x, t) = \frac{2(x + x^2) t^{\frac{1}{2}}}{\sqrt{\pi}}
\]

\[
u_2(x, t) = (1 + 2x)(x + x^2) t
\]

\[
u_3(x, t) = \frac{4(1 + 2x)^2 (x + x^2)}{3\sqrt{\pi}} + \frac{8(x + x^2)^3}{\pi\sqrt{\pi}} t^{\frac{3}{2}}
\]

So the solution by ADM is:

\[
u(x, t) = x + \frac{2(x + x^2) t^{\frac{1}{2}}}{\sqrt{\pi}} + (1 + 2x)(x + x^2) t + \frac{4(1 + 2x)^2 (x + x^2)}{3\sqrt{\pi}} + \frac{8(x + x^2)^3}{\pi\sqrt{\pi}} t^{\frac{3}{2}}
\] (14)
3. Solving Problem by Mittag-Leffler Function

Regard to this paper, we consider the following modified Mittag-Leffler function:

\[ h_p(x) = \sum_{k=1}^{\infty} \frac{x^{kp-1}}{(kp - 1)!} = \frac{x^{p-1}}{(p - 1)!} + \frac{x^{2p-1}}{(2p - 1)!} + \frac{x^{3p-1}}{(3p - 1)!} + \ldots \]  

we can consider the equation (2) as an ordinary deferential equation like:

\[ y^{(\alpha)} = y + y^2, \]  

where \( \alpha \) is the fractional order derivative of \( y \).

To find the approximate solution of (16), we consider the following expressions:

\[ I^\alpha \frac{t^{k\alpha}}{(k\alpha)!} = \frac{t^{(k+1)\alpha}}{(k + 1)!} + c(1 + \alpha) \]

for this the approximate solution of equation (13) can be in the following form:

\[ u(x, t) = \alpha_0(x) + \alpha_1(x) \frac{t^\alpha}{\alpha!} + \alpha_2(x) \frac{t^{2\alpha}}{(2\alpha)!}. \]

Regarding the initial condition \( u(x, 0) = \varphi(x) = \alpha_0(x) \), by using fractional integral \( I^\alpha_t \) from both sides of (2), we have:

\[ u(x, t) = \varphi(x) + I^\alpha(\alpha_0(x) \frac{t^0}{0!} + \alpha_1(x) \frac{t^\alpha}{\alpha!} + \alpha_2(x) \frac{t^{2\alpha}}{(2\alpha)!}) + I^\alpha[\alpha_0^2 + 2\alpha_0(x)\alpha_1(x) \frac{t^\alpha}{\alpha!} + (2\alpha)!((\alpha_1(x))^2 \frac{1}{(\alpha!)^2} + 2\alpha_0(x)\alpha_2(x) \frac{t^{2\alpha}}{(2\alpha)!}) + 2\alpha_1(x)\alpha_2(x) \frac{t^{3\alpha}}{3!(3\alpha)!} + \alpha_2(x)^2 \frac{t^{4\alpha}}{4!(4\alpha)!}], \]

therefore by considering the operator \( I^\alpha \) and according to (17) we have:

\[ u(x, t) = \alpha_0^2 \frac{t^\alpha}{\alpha!} + 2\alpha_0(x)\alpha_1(x) \frac{t^{2\alpha}}{(2\alpha)!} + (\alpha_1(x))^2 \frac{2(\alpha)!}{(\alpha!)^2} + 2\alpha_0(x)\alpha_2(x) \frac{t^{3\alpha}}{(3\alpha)!} + 2\alpha_1(x)\alpha_2(x) \frac{t^{4\alpha}}{3!(2\alpha)!} + \alpha_2(x)^2 \frac{t^{5\alpha}}{(4\alpha)!}], \]

Finally, the following resulted for the unknown coefficients \( \alpha_j(x), j = 0, 1, 2 \) are: \( \alpha_0(x) = \varphi(x), \alpha_1(x) = \alpha_0(x) + \alpha_0(x)^2, \alpha_2(x) = \varphi(x) + \varphi^2(x) + 2\varphi(x)[\varphi(x) + \varphi^2(x)] \alpha_2(x) = \alpha_1(x) + 2\alpha_0(x)\alpha_1(x) \)

Hence the approximate solution \( u(x, t) \) is:

\[ u(x, t) = \varphi(x) + [\varphi(x) + \varphi^2(x)] \frac{t^\alpha}{\alpha!} + (\varphi + \varphi^2)(1 + 2\varphi) \frac{t^{2\alpha}}{(2\alpha)!} \]

if \( \varphi(x) = x, \alpha = \frac{1}{2} \) and \( \Gamma(\alpha + 1) = \alpha! \), the solutions (14), (21) are similar.

References

LINEAR SEISMIC DATA PROCESSING OF AREA OBSERVING SYSTEMS

KABANIKHIN S.I.\(^1,2,3\), NOVIKOV N.S.\(^1,3\), SHISHLENIN M.A.\(^1,2,3\)

1Institute of Computational Mathematics and Mathematical Geophysics SB RAS, Academician Lavrentiev avenue, 6, 630090, Novosibirsk
2Sobolev Institute of Mathematics, Academician Kopyug avenue, 4, 630090, Novosibirsk
3Novosibirsk State University, Pirogova str. 2, Novosibirsk, 630090, Russia
e-mail: kabanikhin@sscc.ru, novikov-1989@yandex.ru, mshishlenin@ngs.ru

1. INTRODUCTION

We consider the solution of the one-dimensional inverse problem for determining the coefficients of the hyperbolic second-order equation that describes the propagation of seismic waves through the medium and our goal is to recover elastic parameters of this medium. The problem of determining the structure of the subsurface by using measurements, obtained on the daylight surface is one of the basic problems in seismology. We propose a method based on the Gelfand-Levitan-Krein approach. This method is originated from the classic works of I.M. Gelfand, B.M. Levitan, M.G. Krein and V.A. Marchenko. During the second half of the XX century, the methods for developed many other researchers and applied for several problems of acoustics, seismics, geoelectrics [1–11]. The essence of the method lies in reduction of the nonlinear inverse problem to a family of linear integral equations. The ability of the method to deal with the nonlinearity of the coefficient inverse problems is important, because nonlinearity is the one of the major difficulties of such problems, after the instability. Another major aspect of the proposed approach is the fact that the method do not uses a priori information, which is important due to the applied nature of the considered problems. Such methods can be used for the obtaining the basic information about the medium. The update of the model parameters can be provided after that by some different and more precise method.

2. RECONSTRUCTION OF S-WAVE VELOCITY AND DENSITY BY GELFAND-LEVITAN-KREIN APPROACH

In this paper we focusing on the problem of recovering the density \(\rho(z)\) and shear wave velocity \(v_s(z)\). We use the approach, proposed in [1]. We consider the inverse problem of theory of elasticity in cylindrical coordinates: \(r = \sqrt{x^2 + y^2}, \theta = \text{arctg}(\frac{y}{x})\). Let us consider, that the stress, applied to the surface \(z = 0\), described by the following:

\[
\sigma_z = 0, \tau_{rz} = 0, \tau_{\theta z} = -\delta(t) \frac{1}{4\pi} \frac{d}{dr} \left[ \delta(r) \right].
\]

In this case, as it is shown in [1], the process of propagation of the SH-waves take place, and the displacement vector \(U_\theta\) contains only one non-zero component \(U_\theta = U_\theta(r, z, t)\), which satisfies the following equation:

\[
\frac{\partial^2 U_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r^2} + \frac{\partial^2 U_\theta}{\partial z^2} + \frac{\mu'(z)}{\mu(z)} \frac{\partial U_\theta}{\partial z} = \frac{\rho(z)}{\mu(z)} \frac{\partial^2 U_\theta}{\partial t^2}
\] (1)
and the following initial and boundary conditions:

$$ U_{\theta}|_{t<0} = 0, \quad \frac{\partial U_{\theta}}{\partial z} = \frac{1}{\mu_0} \sigma(t)b(r). \quad (2) $$

We assume that the data of inverse problem \( \hat{f}(r, t) = U_{\theta}(r, z, t)|_{z=0} \) are known.

Let us apply Hankel transform for the problem (1)-(2) to obtain the following system:

$$ \frac{1}{v_s^2} \frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial \ln(\mu)}{\partial z} \frac{\partial U}{\partial z} - k^2 U; \quad (3) $$

$$ U(z, t; k)|_{t<0} = 0, \quad \frac{\partial U}{\partial z}|_{z=0} = \frac{1}{4\pi\mu_0}\delta(t); \quad (4) $$

$$ U(z, t; k)|_{z=0} = f_k(t). \quad (5) $$

Here \( k \) is the integer parameter, \( \mu_0 \) is the known value of the function \( \mu(z) \) at the surface \( z = 0 \) and \( f_k(t) \) are known functions. The next step is to rewrite the equation (9) for travel-time coordinates by considering the variable

$$ x = \int_0^z \frac{d\xi}{v_s(\xi)} $$

In terms of variables \( x, t \) the problem (3) - (5) can be rewritten as follows:

$$ \frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} - \frac{\sigma'(z)}{\sigma(z)} \frac{\partial U}{\partial x} - k^2 v_s^2 U; \quad (6) $$

$$ U(x, t; k)|_{t<0} = 0, \quad \frac{\partial U}{\partial x}|_{x=0} = \frac{1}{4\pi\sigma_0} \delta(t); \quad (7) $$

$$ U(x, t; k)|_{x=0} = f_k(t). \quad (8) $$

Here \( \sigma(x) = \frac{1}{\sqrt{\rho \mu}} \) is the acoustic impedance of the medium. In case of \( k = 0 \) the equation (6) reduces to the following:

$$ \frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} - \frac{\sigma'(z)}{\sigma(z)} \frac{\partial U}{\partial x} $$

The problem (6)-(8) therefore, reduces to the inverse problem for acoustic equation. We use the dynamic version of the Gelfand-Levitan-Krein approach, considered, for example, in [2], to determine the function \( \sigma(x) \) by given data of inverse problem \( f_0(t) \). The inverse problem is reduced to solving the following family of integral equations:

$$ -2f_0(+0) V(x, t) - \int_{-x}^x V(x, s) f_0'(t - s) ds = 1, \quad t \in (-x, x) \quad (9) $$

For every fixed \( x \) ratio (9) is linear integral equation of the Fredholm type of the second kind. The solution of inverse problem \( \sigma(x) \) is connected with the solution of Krein equation:

$$ \sigma(x) = 0.5V(0, 0)V^{-2}(x, x). \quad (10) $$

The value \( V(0, 0) \) can be calculated by using the given value \( \sigma(0) \) in (10).

The considered approach allows to recover the acoustic impedance \( \sigma(x) \). Now we consider (6) for \( k \neq 0 \) and focusing on calculating shear wave velocity \( v_s(x) \).

Let \( U(x, t) = \hat{U}(x, t)\sqrt{\frac{\sigma(x)}{\sigma(0)}} \) we obtain from (6) the following system:

$$ \frac{\partial^2 \hat{U}}{\partial t^2} = \frac{\partial^2 \hat{U}}{\partial x^2} - q(x; k)\hat{U}; \quad (11) $$

$$ \hat{U}(x, t; k)|_{t<0} = 0, \quad \frac{\partial \hat{U}}{\partial z}|_{z=0} = \delta(t); \quad (12) $$

$$ \hat{U}(x, t; k)|_{x=0} = \hat{f}_k(t). \quad (13) $$
Here
\[ q(x; k) = k^2 v_s^2 - \frac{1}{2} \sigma'' + \frac{3}{4} \left( \frac{\sigma'}{\sigma} \right)^2. \] (14)

Thus, if we calculate \( q(x; k) \) for some specific \( k \neq 0 \), then we can calculate shear wave velocity \( v_s(x) \) from (14) by using known function \( \sigma(x) \).

The system (11)-(13) is equivalent to the following family of integral equations:
\[ \tilde{w}_k(x, t) + \int_{-x}^{x} \tilde{f}'_k(t - \tau) \tilde{w}_k(x, \tau) d\tau = -\frac{1}{2} \left[ \tilde{f}'_k(t - x) + \tilde{f}'_k(t + x) \right], \quad x > 0, t \in (-x, x). \] (15)

Solution of the Gelfand-Levitan equation (15) is connected with the coefficient \( q(x) \):
\[ q(x; k) = 4 \frac{d}{dx} w_k(x, x - 0) \]

Now we can formulate the following procedure for reconstructing shear wave velocity \( v_s \) and the density \( \rho \). First, we solve Krein equation (9), using the inverse problems \( f_0(t) \) and restore the impedance \( \sigma \) using (10). Then we solve Gelvand-Levitan equation (15) and compute function \( q(x) \) by (2). When both functions \( q(x) \) and \( \sigma(x) \) are known, we use the ratio (14) to restore the shear wave velocity \( v_s(x) \). Now we get functions \( \sigma(x) = \sqrt{\frac{\rho}{\mu}} \) and \( v_s(x) = \sqrt{\frac{E}{\rho}} \) and use them to recover density \( \rho(x) \). Moreover, we can use calculated velocity \( v_s(x) \) to reverse travel-time transform and transfer to the original coordinates \( z \).

The work was supported by the RFBR (projects 18-31-00409, 16-29-15120 and 16-01-00755).

**Keywords:** Gelfand-Levitan-Krein equation, numerical method, coefficient inverse problem.

**AMS Subject Classification:** 31A25, 49N45, 65M32, 65R32.

**References**


NEW FORMULATIONS FOR THE TRAVELING REPAIRMAN PROBLEM WITH TIME WINDOWS

IMDAT KARA¹, GOZDE ONDER UZUN¹

¹Baskent University, Department of Industrial Engineering, Ankara, Turkey  
e-mail: ikara@baskent.edu.tr, gonder@baskent.edu.tr

The Traveling Salesman Problem (TSP) is the basis of the many routing problems. In a routing problem, waiting time or latency of a customer is defined as the time passed from the beginning of the travel until this customer’s service is completed. The Minimum Latency Problem is to find a tour starting from a depot and visiting all nodes in such a way that the total latency is minimized [2]. This problem is also known as the Delivery Man Problem [5] or the Traveling Repairman Problem (TRP) [12]. It has been shown that TRP is NP-hard [12]. Therefore, solution strategies for TRP and its variants are concentrated on the exact solution procedures and heuristics [11]. There are a few formulations for TRP and its variants that can be used directly by an optimizer.

In routing problems, time window for a city (customer, client) is an interval defined by the earliest and the latest times that this city can be visited. The TRP is called TRP with time windows (TRPTW) which can be stated simply as finding the optimal route (minimizing total latency) of a traveller visiting a set of cities (nodes) exactly once, where each city must be visited within a given time window. The first paper appeared in the literature on TRPTW in 1992 [12], where the problem has been defined and polyhedral analysis conducted and solution strategies are discussed, any mathematical models are not proposed. TRPTW arises in many real-life problems such as school bus routing, transportation of perishables goods, scheduling problems in manufacturing, nurse serving problem in hospital and etc. [12]. A polynomial size mixed integer programming formulation for TRPTW proposed by Heilporn et al. [6]. They tried to solve some problems directly and observe that the largest problem size solved within 3600 seconds was 15 nodes. They propose two heuristics for solving TRPTW. As far as we are aware, this is the only one mathematical model proposed for directly TRPTW. We named this formulation as existing formulation one (EF1).

One of the most important variants of the TRP is multiple TRP where there are k travellers (kTRP) [4]. Kara et al. [9] overviewed and conducted a computational analysis on the formulations for kTRP. There are a few studies on the kTRPTW. kTRPTW is defined and a polynomial size formulation is presented by R. V. der Meer in 2000 [10]. This formulation becomes a formulation for TRPTW when the number of the travellers taken as one. Recently, a mixed integer linear programming model is proposed for the case where travel times between nodes depends upon the travellers [1]. They named the problem as heterogeneous TRPTW (hetTRPTW), they used only earliest time for all customers [1]. If the heterogeneity is neglected, this formulation reduces to the earlier homogeneous TRPTW formulation [10]. So, it can be said that, there is only one basic formulation for TRPTW with multiple repairmen which can also be considered a mathematical model for TRPTW. We handled this formulation with single traveller as the second existing formulation for TRPTW and named it as EF2.
So, there exist only two formulations for finding the optimal solution of the TRPTW directly in the literature. The emerging developments in the information technologies and falling price in software and hardware allow us to find optimal solution of moderate size real life problems directly by using a suitable software and user-friendly formulations. Also, mathematical formulations facilitate to add additional decision maker’s restrictions can be taken as new constraints of the model. Mathematical formulation allow us to make sensitivity analysis and to handle some extensions of the problem by adapting the model to new situations. These developments and advantages of formulations motivate us to develop new mathematical models for the TRPTW.

In this paper, we propose four new mathematical modes for TRPTW by naming them as NF1, NF2, NF3 and NF4. In NF1, NF3 and NF4 latencies are defined on the nodes and in NF2 latency is defined on the arcs of the underlying graph. The constraint set of NF1 is mostly based on the formulation given for TSPTW by Kara et al. [8] and NF4 is based on the formulation given by Kara and Derya [7]. We present four new mixed integer linear programming formulation for TRPTW with $O(n^2)$ binary decision variables and $O(n^2)$ constraints. In order to see computational performances of the new and existing formulations, we conduct a detailed computational analysis by solving instances up to 100 nodes appeared in the literature. We used the benchmark instances of Dumas et al. [3]. They include 50 test problems with 20, 40, 60, 80 and 100 nodes and we used 10 problem sets for each node number with time-window widths ranging from 20 to 40 time units. All problems are solved with CPLEX 12.5.0.1 by using Intel Core i7-3630QM CPU 2.40 GHz and 16 GB RAM computer. The upper time limit is taken as 3600 seconds for all computations. Computational comparisons are focused on CPU times. The CPU times of optimal values obtained in time limit are used to calculate average CPU times.

The results of computational analysis are summarized in Table 1. Average values of any group of the problem with all formulations do not computed when there is at least one problem that can not be solved optimality within 3600 seconds. We observe that, the existing formulation EF1 solved the problems up to 20 nodes, the second existing formulation EF2 and the new formulation NF1 solved the problems up to 40 nodes with 40 width and 60 nodes with 20 width. NF2 solved up to 40 nodes. NF3 and NF4 solved up to 100 nodes, optimally. We do not observe any significant differences between CPU times of NF3 and NF4. We gave the average computational results up to 100 nodes in Table 1. Both formulations are capable to solve the problems up to 100 nodes within seconds. We computationally show that the performances of proposed formulations NF3 and NF4, in terms of CPU times are extremely better than the existing ones. As it is seen in Table 1, width of the time windows effects solution times directly. When time windows become enlarger CPU times rapidly increases.

Our contributions may be summarized as follows:

- Existing formulations of the TRPTW and its variants are reviewed.
- Four new easy to use formulations for TRPTW are developed.
- Performance analysis of the existing and proposed formations are evaluated.
- Formulations that can be directly usable for solving small and moderate size problems are proposed.

Adaptation of the proposed formulations to the variants of the TRP, especially kTRPTW are under considerations.
Table 1. Average CPU values of problem sets for all formulations

<table>
<thead>
<tr>
<th>Problem Number of Nodes</th>
<th>Width of Time Window</th>
<th>EF1</th>
<th>EF2</th>
<th>NF1</th>
<th>NF2</th>
<th>NF3</th>
<th>NF4</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>20</td>
<td>45.13</td>
<td>0.035</td>
<td>0.052</td>
<td>0.873</td>
<td>0.018</td>
<td>0.023</td>
</tr>
<tr>
<td>20</td>
<td>40</td>
<td>394.882</td>
<td>0.178</td>
<td>0.246</td>
<td>4.00</td>
<td>0.038</td>
<td>0.038</td>
</tr>
<tr>
<td>40</td>
<td>20</td>
<td>-</td>
<td>0.41</td>
<td>0.777</td>
<td>43.197</td>
<td>0.067</td>
<td>0.063</td>
</tr>
<tr>
<td>40</td>
<td>40</td>
<td>-</td>
<td>48.172</td>
<td>88.402</td>
<td>1788.428</td>
<td>0.138</td>
<td>0.15</td>
</tr>
<tr>
<td>60</td>
<td>20</td>
<td>-</td>
<td>3.47</td>
<td>17.01</td>
<td>-</td>
<td>0.113</td>
<td>0.073</td>
</tr>
<tr>
<td>60</td>
<td>40</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>3.822</td>
<td>3.096</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>20</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>3.822</td>
<td>3.096</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>40</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>48.388</td>
<td>70.544</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>20</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.325</td>
<td>0.235</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>40</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>788.893</td>
<td>1011.666</td>
<td></td>
</tr>
</tbody>
</table>

Keywords: Traveling repairman, minimum latency, routing, time windows, modeling.

AMS Subject Classification: 90-08, 65K05.

REFERENCES

AUGMENTED LAGRANGIAN BASED SOLUTION METHOD FOR ONE-DIMENSIONAL CUTTING STOCK AND ASSORTMENT PROBLEM

NERGIS KASIMBEYLI1, TUGBA SARAC2, REFAIL KASIMBEYLI3

1Department of Industrial Engineering, Anadolu University, Turkey
2Industrial Engineering Department, Eskisehir Osmangazi University, Turkey
3Department of Industrial Engineering, Anadolu University, Turkey
e-mail: nkasimbeyli@anadolu.edu.tr

Abstract. This paper is a continuation of the research on cutting problems given in [2]. The paper presents a mathematical model for a one-dimensional cutting stock and assortment problem and an augmented Lagrangian based solution method. We propose a new solution algorithm for solving the presented model. The performance of the solution approach is demonstrated on test problems.

Keywords: Integer programming, one-dimensional assortment problem, cutting stock problem, stock size selection, trim loss minimization, sharp augmented Lagrangian.

AMS Subject Classification: 90B50, 90B80, 90C27, 90C29, 90C59, 90C90, 97M40.

1. Mathematical model

During a planning period, \( n \) order pieces are to be fulfilled. The sizes and the number of pieces are given. The assortment problem involves the choice of the best combination of different types of standard lengths to be maintained as inventory and to be used for cutting the order pieces. In this case, the problem is to select the optimal (minimal) number of roll sizes have to be stocked and to find the corresponding cutting patterns in order to produce the required order pieces.

Notations:

2. Sets and parameters

Let

- \( m \) be the number of all roll sizes (standard lengths), which are available for the producer,
- \( n \) be the number of all order pieces (products) with different widths to be fulfilled during the planning period,
- \( I = \{1, \ldots, m\} \) be a set of roll sizes,
- \( J = \{1, \ldots, n\} \) be a set of order pieces,
- \( c_j \) be the length of order piece \( j \),
- \( d_j \) be the demand for order piece \( j \),
- \( L_i \) be the length of roll \( i \),
- \( K_i \) be the maximal number of times the roll of type \( i \) can be used in the production process,
- \( S \) be the total number of roll types used.
3. Decision variables

Let

- $x_{ik}$ be the binary variable indicating whether the roll of type $i$ will be used for $k^{th}$ time:
  \[ x_{ik} = \begin{cases} 1 & \text{if roll } i \text{ is used for } k^{th} \text{ time}, \\ 0 & \text{otherwise.} \end{cases} \quad (1) \]

- $z_i$ be the binary variable indicating whether the roll of type $i$ will be used or not:
  \[ z_i = \begin{cases} 1 & \text{if roll } i \text{ is used}, \\ 0 & \text{otherwise.} \end{cases} \quad (2) \]

- $y_{ijk}$ be the number of times the order piece $j$ is involved in the roll of type $i$ when this roll is used for $k^{th}$ time.

Under these notifications we can formulate an IP model $(P)$ for the one-dimensional cutting stock and assortment problem described above, in the following form:

\[
(P) \quad \min f(x, y, z) = \sum_{i=1}^{m} \sum_{k=1}^{K_i} [L_i x_{ik} - \sum_{j=1}^{n} c_j y_{ijk}] \quad (3)
\]

subject to

\[
\sum_{i=1}^{m} \sum_{k=1}^{K_i} y_{ijk} = d_j \text{ for all } j = 1, \ldots, n, \quad (4)
\]

\[
\sum_{j=1}^{n} c_j y_{ijk} \leq L_i x_{ik} \text{ for all } i = 1, \ldots, m; k = 1, \ldots K_i, \quad (5)
\]

\[
z_i \leq \sum_{k=1}^{K_i} x_{ik} \leq K_i z_i \quad i = 1, \ldots, m, \quad (6)
\]

\[
\sum_{i=1}^{m} z_i \leq S, \quad (7)
\]

$x_{ik}$ and $z_i$ binary, $y_{ijk}$ nonnegative integer for all

\[ i = 1, \ldots, m, \quad j = 1, \ldots, n, \quad \text{and } k = 1, \ldots K_i. \]

Constraint set (4) ensures that the demand for any order piece has to be met. Constraint set (5) are called the knapsack constraints and ensures that the length of the cutting pattern $k$ generated for the roll of type $i$ can not exceed the lengths $L_i$ of this roll. If any cutting pattern is assigned to a roll of type $i$, then constraint set (6) forces $x_{ik} = 1$ for some $k$. If no cutting pattern is assigned to a roll of type $i$, then these constraints force $x_{ik} = 0$ for all $k$. The constraint set (7) restricts the total number of roll types used. Finally, the integrality constraints are added to functional constraints.

Note that the problem $(P)$ defined by relations (3)-(7) together with the integrality constraints, may involve a huge number of binary and integer decision variables which causes difficulties in the solution process.
4. Solution approach

Lagrange functions play a key role in mathematical programming. Classical Lagrange functions and algorithms based on them can only be applied to some special classes of constrained optimization problems. The use of the augmented and sharp augmented Lagrangians lead to the design of efficient algorithms for solving a broad class of constrained optimization problems without convexity conditions. Sharp augmented Lagrangians were studied in detail in [1] and [3]. Now we briefly describe them for the following optimization problem:

\[(P) \text{ minimize } f_0(x) \text{ subject to } f(x) = 0, \ x \in X,\]

where \(X \subset \mathbb{R}^n\) is a compact set and functions \(f_0 : \mathbb{R}^n \to \mathbb{R}, \ f : \mathbb{R}^n \to \mathbb{R}^p\) are continuous. Denote by \(R_+\) the set of nonnegative numbers. The sharp augmented Lagrangian \(L : \mathbb{R}^n \times \mathbb{R}^p \times R_+ \to \mathbb{R}\) associated with (8) is defined as follows:

\[L(x, u, c) = f_0(x) + c\|f(x)\| - \langle u, f(x) \rangle,\]

where \(x \in \mathbb{R}^n, u \in \mathbb{R}^p\) and \(c \in R_+\). The solution set of problem (8) is denoted by \(\text{Sol}(P)\). We typically denote an element of \(\text{Sol}(P)\) by \(\bar{x}\). The dual function \(H : \mathbb{R}^p \times R_+ \to \mathbb{R}\) is defined as:

\[H(u, c) = \min_{x \in X} [f_0(x) + c\|f(x)\| - \langle u, f(x) \rangle].\]

Then the dual problem of (8) is given by

\[\left(\overset{\star}{P}\right) \text{ maximize } H(u, c) \text{ subject to } u \in \mathbb{R}^p, \ c \in R_+.\]

The solution set of problem (11) is denoted by \(\text{Sol}(\overset{\star}{P})\). We denote an element of \(\text{Sol}(\overset{\star}{P})\) by \(\bar{z} = (\bar{u}, \bar{c})\). Consider the set

\[X(u, c) = \text{Argmin}_{x \in X} [f_0(x) + c\|f(x)\| - \langle u, f(x) \rangle].\]

In this paper, we present a method for solving the mathematical model presented. This method is based on sequential minimization of the augmented Lagrangian function for selected Lagrange multipliers and constraints. The constraints will be included in the function \(f\) step by step by checking the optimality criterion. Once the optimality criteria is satisfied, the algorithm stops.

REFERENCES

OPTIMAL CONTROL OF A GIVING UP SMOKING MODEL WITH AGE-STRUCTURED IN SMOKING CLASSES

ASAF KHAN$^{1,2}$, GUL ZAMAN$^2$

$^1$Department of Mathematics and Statistics, University of Swat, Khyber Pakhtunkhwa, Pakistan
$^2$Department of Mathematics, University of Malakand, Khyber Pakhtunkhwa, Pakistan
e-mail: asafkhan@uswat.edu.pk, gzaman@uom.edu.pk

1. INTRODUCTION

Tobacco use has many health related, economic, social, and environmental consequences and hence it is a major barrier to sustainable development. It has been the primary universal source of preventable death. Tobacco use kills above seven millions people annually, where over eighty percent contribution to this figure is from low and middle-income countries [10]. Though passive smoking also kills, but this rate of killing is less. It is observed that tobacco use will kill one billion or more people in the current century unless urgent action is taken [9]. In order to understand smoking dynamics and to find an effective control strategy, numerous researchers from different fields and organizations have presented some solid work such as [8, 12]. For obtaining a smoking-free environment, mathematicians used the tools of mathematical modeling and optimal control theory. The first mathematical model for giving up smoking was presented by Castillo-Garsow et al. [1] in 2000. This work opened a new channel of research for mathematicians in the field of smoking dynamics and control. Sharami and Gumel presented a model [5] by considering chain and mild smokers and gave a detail analysis therein. Zaman extended the work of Castillo-Garsow et al. by taking into account the occasional smokers [11]. Numerous other researchers developed and analyzed giving up smoking models such as [2, 13, 14] in addition to his work.

The above mentioned authors only considered time as independent variable and they assumed that the dynamics of different classes may change only with the passage of time. However, there are many other factors such as age of initiation, time-since-smoking, and degree of inhalation etc that could affect the dynamics of smoking [7]. A person whose time-since-smoking is 10-15 will face many health complications compared to those who started smoking a year ago. A recent study by Rahman et al. [4] included the time-since-smoking only in smoker’s class. The authors developed the model and presented a detail analysis on stability and control therein. A similar model but from a different scenario was constructed and analyzed by Zeb et al. in [15]. However, it is interesting to see that time-since-smoking is equally important for occasional smokers as well. Since, a person who starts smoking occasionally and his time-since-smoking become longer and longer; ultimately, he will also become at risk to different diseases. Also, time-since-infection will decide which occasional smoker will come to the smokers class. Therefore, in order to understand a detailed dynamics of giving up smoking model which reflects the cumulative hazards of smoking and near to reality, one must include time-since-smoking both in occasional smokers and smokers classes.

The first application of optimal control theory to giving up smoking models is credited to Zaman [12] who used optimal control theory for minimizing the number of light and persistent
smokers and maximizing the number of quit smokers. The author used two types of control variable; education and treatment campaigns for light smokers and smokers, respectively. Numerous other researchers used different control measures in order to obtain the desired goal (see [4,14]).

Statistics shows that people are most likely to begin the use of tobacco as adolescents, it is especially important to inform young people about the harms of tobacco use before they start. Education campaign is more likely to prevent adolescents from initiating smoking also it will compel occasional smokers to think about cutting down or quitting [6]. Most smokers who are aware of the dangers of tobacco and want to quit are facing difficulties in quitting because of the highly addictive nature of nicotine. In case where education campaign does not work for smokers, one must also include treatment with nicotine replacement therapy (NRT) and other drugs that require a prescription [9]. Pharmacological therapy can double or triple quitting rates [3,9].

Based on these evidences, we will extend the work of Rahman et al by introducing time-since-smoking both in occasional smokers and smokers classes. Once the model is developed, we will show that their exist an optimal control problem. Furthermore, since education campaign does not work for smokers, we will use education campaign and anti-nicotine drugs for potential smokers and smokers, respectively, as control variables.

2. The model

In order to construct the model, we introduce another independent variable a called time-since-smoking (or smoking-age), which denotes the time that has been passed since smoking. The entire population is stratified into four subclasses, potential smokers $P(t)$, occasional smokers $L(t)$, smokers $S(t)$ and quit smokers $Q(t)$. We express the number of occasional smokers at time $t$ as $L(t) = \int_0^\infty l(t,a)\,da$ where $l(t,a)$ is the density of social smokers and similarly for the number of smokers $S(t) = \int_0^\infty s(t,a)\,da$. Keeping in view some assumptions, our optimal control problem which minimize the objective functional, is given by

$$J(u) = \int_0^T (P(t) + L(t) + S(t) + \frac{B_1}{2} u_1^2(t) + \frac{B_2}{2} u_2^2(t))\,dt,$$

subject to

$$\frac{dP(t)}{dt} = \Lambda - \nu P(t) - P(t) \int_0^\infty \sigma(a)s(t,a)\,da - u_1(t)P(t), \quad (2a)$$

$$\frac{\partial l(t,a)}{\partial t} + \frac{\partial l(t,a)}{\partial a} = - (\nu + \xi(a)l(t,a), \quad (2b)$$

$$\frac{\partial s(t,a)}{\partial t} + \frac{\partial s(t,a)}{\partial a} = - (\mu + \nu - u_2(t))s(t,a), \quad (2c)$$

$$\frac{dQ(t)}{dt} = (\mu + u_2(t))S(t) - \nu Q(t) + u_1(t)P(t), \quad (2d)$$

$$l(t,0) = P(t) \int_0^\infty \sigma(a)s(t,a)\,da, \quad s(t,0) = \int_0^\infty \xi(a)l(a,t)\,da, \quad (2e)$$

$$L(t) = \int_0^\infty l(t,a)\,da, \quad S(t) = \int_0^\infty s(t,a)\,da, \quad (2f)$$

$$l(0,a) = l_0(a), \quad s(0,a) = s_0(a), \quad P(0) = P_0 > 0, \quad Q(0) = Q_0 \geq 0. \quad (2g)$$

Here $B_i, i = 1, 2$ are positive constants (called balance factors) which represents an individual (potential and smoker) level of acceptance of the controls. We considered two control variables $u_1(t), u_2(t) \in U$ relative to the state variables $P(t), l(t,a), s(t,a), Q(t)$, where $U = \left\{ u_1, u_2 \in L^2(0,T)|0 \leq u_1(t) \leq c_1 \leq 1, 0 \leq u_2(t) \leq c_2 \leq 1, t \in [0,T] \right\}$, says an admissible control set.
Physical interpretation of the control variables are; the control $u_1(t)$ represent the education campaign which is to be implemented on potential smokers only, while $u_2(t)$ stand for anti-nicotine drugs which is to be given to smokers class. Our goal in the current optimization problem is to attain the minimum level of potential smokers, occasional smokers and smokers population and to maximize the number of quit smokers.

3. Main results

**Theorem 1.** There exists optimal control variables $u_1^*, u_2^* \in U$ such that $J(u_1^*, u_2^*) = \min_{u_1, u_2 \in U} J(u_1, u_2)$, subject to the control system (2).

**Theorem 2.** If $u_1^*, u_2^* \in U$ is an optimal control which minimizing (1) and $(P^*(t), L^*(t), S^*(t), Q^*(t))$ and $(\lambda_1(t), \lambda_2(t, a), \lambda_3(t, a), \lambda_4(t))$ are the corresponding state and adjoint variables, respectively, then

\[
u_1^*(t) = \min \left\{ l_1, \max \left\{ 0, \frac{(\lambda_1(t) - \lambda_4(t)) P^*(t)}{B_1} \right\} \right\},
\]

\[
u_2^*(t) = \min \left\{ l_2, \max \left\{ 0, \int_0^\infty s(t, a) \left[ \lambda_3(t, a) - \lambda_4(t) \right] da \right\} \right\}.
\]

**Keywords:** Optimal control, age-structure, gâteaux derivative.

**AMS Subject Classification:** 00A72, 49J20, 49J50.

**References**


ON THE SOLUTION OF ONE PROBLEM FOR LINEAR HYPERBOLIC TYPE LOADED DIFFERENTIAL EQUATION BY THE METHOD OF FINITE DIFFERENCES

Z.F. KHANKISHIYEV

1Baku State University, Baku, Azerbaijan
E-mail: hankishiyev.zf@yandex.com

1. STATEMENT OF THE PROBLEM

Suppose, it is required to find a continuous function \( u = u(x, t) \) in the closed domain \( D = \{0 \leq x \leq l, 0 \leq t \leq T\} \) that satisfies equation

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}(k(x,t)\frac{\partial u}{\partial x}) + b(x)u(x,t) + \sum_{k=1}^{m} d_k(x)u(x,\bar{t}_k) + f(x,t), \quad 0 < x < l, \quad 0 < t \leq T, \tag{1}
\]

boundary conditions

\[
\frac{\partial u(0,t)}{\partial x} + \alpha_1 u(0,t) + \beta_1 u(l,t) = \mu_1(t), \quad 0 \leq t \leq T, \tag{2}
\]

and initial conditions

\[
u(x,0) = \varphi_1(x), \quad \frac{\partial u(x,0)}{\partial t} = \varphi_2(x), \quad 0 \leq x \leq l, \tag{3}\]

where \( k(x,t) > 0, b(x), d_k(x), k = 1, 2, \ldots, m, f(x,t), \mu_1(t), \mu_2(t) \) are known continuous functions of their arguments, \( \alpha_1, \beta_1, \alpha_2, \beta_2 \) are given real numbers, and \( \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_m \in (0, T) \)-fixed points. It is assumed that problem (1)-(3) has a unique solution that possesses the derivatives required in the course of the exposition.

2. THE DIFFERENCE PROBLEM

The segment \([0, l]\) of the axis is divided by points \( x_n = nh, n = 0, 1, 2, \ldots, N, h = l/N, \) into equal parts, and the segment \([0, T]\) of the axis \( Ot \) by dots \( t_j = j\tau, \) \( j = 0, 1, 2, \ldots, j_0, \tau = T/j_0, \) into \( j_0 \) equal parts. Step \( \tau \) is chosen so that the points \( t_k, \) \( k = 1, 2, \ldots, m, \) are among the points \( t_j = j\tau, j = 1, 2, \ldots, j_0. \) Suppose, that \( t_k = t_{jk}, k = 1, 2, \ldots, m, t_j < t_{j_1} < \ldots < t_{j_m}. \)

Let a grid \( \bar{\omega}_{hr} = \{(x_n, t_j), n = 0, 1, 2, \ldots, N, \quad j = 0, 1, 2, \ldots, j_0\} \) be defined in the domain \( \bar{D}. \) In this grid region \( \bar{\omega}_{hr}, \) problem (1)-(3) is comparable to the following difference problem [1]:

\[
(y_n^j)_{tt} = \Lambda(t_j) (\sigma y_n^{j+1} + (1 - 2\sigma)y_n^j + \sigma y_n^{j-1}) + b(x_n)y_n^j + \sum_{k=1}^{m} d_k(x_n)y_n^{\bar{t}_k} + f_n^j, \quad n = 1, 2, \ldots, N - 1, \quad j = 0, 1, 2, \ldots, j_0 - 1, \tag{4}
\]

\[
\frac{y_n^{j+1} - y_n^j}{\tau^2} + \alpha_1 \frac{y_n^{j+1} + y_n^j}{2} + \beta_1 \frac{y_n^{j+1} + y_n^{j-1}}{2} = \mu_1^j, \quad j = 0, 1, \ldots, j_0 - 1, \tag{5}
\]

\[
\frac{y_n^{j+1} - y_n^j}{\tau} + \alpha_2 \frac{y_n^{j+1} + y_n^j}{2} + \beta_2 \frac{y_n^{j+1} + y_n^{j-1}}{2} = \mu_2^j, \quad j = 0, 1, \ldots, j_0 - 1, \tag{5}
\]

\[
y_n^0 = \varphi_1(x_n), \quad \frac{y_n^1 - y_n^0}{\tau} = \varphi_2(x_n), \quad n = 0, 1, 2, \ldots, N. \tag{6}
\]
Here \( \sigma \) -is real parameter,
\[
(y_n^j)_{tt} = \frac{y_n^{j+1} - 2y_n^j + y_n^{j-1}}{\tau^2}, \quad \Lambda(t_j)y_n^j = (k(x_{n-1/2}, t_j) \cdot y_{n, 2}^j)_x, \quad x_{n-1/2} = x_n - \frac{1}{2}h,
\]
where
\[
\mu_1^j = \mu_1(t_j), \quad \mu_2^j = \mu_2(t_j), \quad \ell_j = \frac{t_j + t_{j+1}}{2}, \quad f_n^j = f(x_n, t_j), \quad \varphi_2(x_n) = \varphi_2(x_n) + \nonumber
\]
\[
+ \tau \cdot \frac{d}{dx} [k(x, 0) \cdot \varphi_1'(x)]_{x=x_n} + f(x_n, 0).
\]

This difference problem approximates the problem (1) - (3) with accuracy \( O(h^2 + \tau^2) \), if the solution of equation (1) is the function \( u(x, t) \) has bounded partial derivatives with respect to \( x \) and \( t \) up to fourth order in the domain \( D = \{0 < x < l, \quad 0 < t \leq T\} \), and bounded mixed derivatives \( \frac{\partial^{k+1} u}{\partial x^k \partial t^l} \), \( k = 1, 2 \), and the coefficient \( k(x, t) \) has bounded partial derivatives \( \frac{\partial^s k}{\partial x^s} \), \( s = 1, 2 \).

Using equations
\[
y_n^{j+1} = y_n^j + \tau (y_n^j)_t + \frac{\tau^2}{2} (y_n^j)_{tt}, \quad y_n^{j-1} = y_n^j - \tau (y_n^j)_t + \frac{\tau^2}{2} (y_n^j)_{tt},
\]
where
\[
(y_n^j)_t = \frac{y_n^{j+1} - y_n^j}{2\tau}, \quad \text{the expression } \sigma y^{j+1}_n + (1 - 2\sigma)y^{j}_n + \sigma y^{j-1}_n \text{ can be represented in the following form:}
\]
\[
\sigma y^{j+1}_n + (1 - 2\sigma)y^{j}_n + \sigma y^{j-1}_n = \sigma \tau^2(y_n^j)_{tt} + y_n^j.
\]

Taking into account this equality, the difference problem (4) - (6) can be rewritten in the form
\[
(1 - \sigma \tau^2 \Lambda(t_j)) (y_n^j)_{tt} = \Lambda(t_j)y_n^j + b(x_n) y_n^j + \sum_{k=1}^{m} d_k(x_n) y_n^j + f_n^j,
\]

\[
\text{where } n = 1, 2, ..., N - 1, \quad j = 0, 1, 2, ..., j_0 - 1, \quad (7)
\]
\[
\frac{y_0^{j+1} - y_0^j}{\tau} + \alpha_1 y_0^{j+1} + \beta_1 y_0^{j+1} + y_0^j = \mu_1^j, \quad j = 0, 1, ..., j_0 - 1, \quad (8)
\]
\[
y_0^0 = \varphi_1(x_n), \quad (9)
\]

3. Solution of the difference problem

Knowing \( y_0^j \) and \( y_N^j \), from equations (8), one can successively find \( y_0^{j+1} \) and \( y_N^{j+1}, j = 0, 1, ..., j_0 - 1 \). Therefore it is assumed that \( y_0^{j+1} \) and \( y_N^{j+1}, j = 0, 1, ..., j_0 - 1 \), are known. Suppose, that \( y_0^{j+1} = \gamma_{j+1}, \quad y_N^{j+1} = \delta_{j+1}, \quad j = 0, 1, ..., j_0 - 1 \).

Then the difference problem (7) - (9) can be rewritten in the form
\[
(1 - \sigma \tau^2 \Lambda(t_j)) (y_n^j)_{tt} = \Lambda(t_j)y_n^j + b(x_n) y_n^j + \sum_{k=1}^{m} d_k(x_n) y_n^j + f_n^j,
\]

\[
\text{where } n = 1, 2, ..., N - 1, \quad j = 0, 1, 2, ..., j_0 - 1, \quad \gamma_{j+1}, \quad \delta_{j+1}, \quad j = 0, 1, ..., j_0 - 1, \quad (10)
\]
\[
y_0^0 = \varphi_1(x_n) + \sigma \varphi_2(x_n), \quad (11)
\]

knowing \( \Lambda(t_j)y_n^j \) is defined by equality \( \Lambda(t_j)y_n^j = \left( k(x_{n-1/2}, t_j) \cdot y_{n, 2}^j \right)_x \), then after elementary transformations the difference equations (7) can be written in the form
\[
\frac{\sigma \tau^2}{h^2} k(x_{n-1/2}, t_j) y_n^{j+1}_{n-1} + \left( 1 + \frac{\sigma \tau^2}{h^2} (k(x_{n+1/2}, t_j) + k(x_{n-1/2}, t_j)) \right) y_n^{j+1}_n - \frac{\sigma \tau^2}{h^2} k(x_{n+1/2}, t_j) y_n^{j+1}_n + \nonumber
\]
\[
+ \left( \frac{2\sigma - 1}{h^2} \right) k(x_{n-1/2}, t_j) y_n^{j+1}_{n-1} - \left( 2 + \tau^2 b(x_n) + \frac{(2\sigma - 1) \tau^2}{h^2} (k(x_{n+1/2}, t_j) + k(x_{n-1/2}, t_j)) \right) .
\]
\[ y^j_n + \frac{(2\sigma - 1)\tau^2}{h^2} k(x_{n+1/2}, t_j)y^j_{n+1} - \frac{\sigma\tau^2}{h^2} k(x_{n-1/2}, t_j)y^j_{n-1} + \left( 1 + \frac{\tau^2}{h^2} (k(x_{n+1/2}, t_j) + k(x_{n-1/2}, t_j)) \right)y^{j-1}_n - \frac{\sigma\tau^2}{h^2} k(x_{n+1/2}, t_j)y^{j-1}_{n+1} - \tau^2 \sum_{k=1}^m d_k(x_n)y^j_k = \tau^2 f_n^j, \quad n = 1, 2, ..., N - 1, \quad j = 1, 2, ..., j_0 - 1. \] (12)

Thus, the solution of problem (4)-(6) is reduced to the solution of the difference problem (8)-(10).

Rewrite this difference problem in the following matrix form:

\[ A^j y^{j+1} + B^j y^j + A^j y^{j-1} + D_1 y^{j_1} + D_2 y^{j_2} + ... + D_m y^{j_m} = F^j, \quad j = 0, 1, ..., j_0 - 1, \] (13)

\[ y^0 = \varphi_1, \quad y^1 = \varphi_2, \] (14)

where \( A^j, B^j, D_k, k = 1, 2, ..., m \) - known matrices, \( y^j \) - unknown vector, \( F^j \) - known column vector.

Rewrite the difference equations (13) taking into account conditions (14) for each value of \( j \) separately:

\[ A^1 y^2 + D_1 y^{j_1} + D_2 y^{j_2} + ... + D_m y^{j_m} = F^1 - B^1 \varphi_2 - A^1 \varphi_1, \]

\[ A^2 y^3 + B^2 y^2 + D_1 y^{j_1} + D_2 y^{j_2} + ... + D_m y^{j_m} = F^2 - A^2 \varphi_2, \]

\[ A^3 y^4 + B^3 y^3 + A^{j_0-1} y^{j_0-1} + A^{j_0-1} y^{j_0-2} + D_1 y^{j_1} + D_2 y^{j_2} + ... + D_m y^{j_m} = F^{j_0-1}. \] (15)

The matrices \( A^j, j = 1, 2, ..., j_0 - 1 \), have inverse matrices, because they have a diagonal predominance. Therefore, multiplying both sides of the first equation in (15) from the left by \((A^1)^{-1}\), the second equation by \((A^2)^{-1}\) and etc., the last equation by \((A^{j_0-1})^{-1}\) is obtained:

\[ y^2 + (A^1)^{-1} (D_1 y^{j_1} + D_2 y^{j_2} + ... + D_m y^{j_m}) = (A^1)^{-1} (F^1 - B^1 \varphi_2 - A^1 \varphi_1), \]

\[ y^3 + (A^2)^{-1} B^2 y^2 + (A^2)^{-1} (D_1 y^{j_1} + D_2 y^{j_2} + ... + D_m y^{j_m}) = (A^2)^{-1} (F^2 - A^2 \varphi_2), \]

\[ y^{j_0} + (A^{j_0-1})^{-1} B^{j_0-1} y^{j_0-1} + y^{j_0-2} + (A^{j_0-1})^{-1} (D_1 y^{j_1} + D_2 y^{j_2} + ... + D_m y^{j_m}) = (A^{j_0-1})^{-1} F^{j_0-1}. \] (16)

From the first equation in (16) it is possible to determine \( y^2 \), taking into account the found expression for \( y^2 \), from the second equation it is possible to determine \( y^3 \), taking into account the found expressions for \( y^2 \) and \( y^3 \) from the third equation to determine \( y^4 \), etc., from the last one with help to sum \( D_1 y^{j_1} + D_2 y^{j_2} + ... + D_m y^{j_m} \) to determine \( y^{j_0} \).

Separating from the obtained equalities equalities \( j = j_1, j_2, ..., j_m \), system of \( m \) vector equations is obtained. From this system of vector equations can be found \( D_1 y^{j_1} + D_2 y^{j_2} + ... + D_m y^{j_m} \). To do this, it suffices to add these equations, previously multiplied on the left by \( D_1, D_2, \) etc., \( D_m \) correspondingly, and from the obtained equation determine this sum.

Let it be found \( D_1 y^{j_1} + D_2 y^{j_2} + ... + D_m y^{j_m} \). Then, taking into account the value of this expression in the right-hand sides of equalities (16), we can determine \( y^2, y^3, ..., y^{j_0} \).

**Keywords:** Schrodinger equation, finite-difference equations, spectral problem.

**AMS Subject Classification:** 35Q55, 35P30, 47J10, 81Q05.

**References**

SOME INCONSISTENCIES OF FAMILIAR QUANTUM MECHANICAL RELATIONS IN CASE OF SINGULAR POTENTIALS AND OPERATORS

A. KHELASHVILI\textsuperscript{1}, T. NADAREISHVILI\textsuperscript{2}

\textsuperscript{1}St. Andrea the First-called Georgian University of Patriarchy of Georgia, Tbilisi, Georgia, Institute of High Energy Physics
\textsuperscript{2}University Javakhishvili Tbilisi State University, Tbilisi, Georgia
e-mail: anzor.khelashvili@tsu.ge

Abstract. It is shown that a lot of principal theorems in quantum mechanics suffer changes in spherical coordinates when singular operators are considered.

Keywords: General quantum mechanics, singular potentials and operators.

AMS Subject Classification: 81S05, 81Q65.

1. Introduction

In textbooks on quantum mechanics main formulations are concerned to the one dimensional problems and in these cases, as a rule, the wave functions chosen decrease at infinity (Hilbert space). It is clear that when the system is located in finite volume, the inclusion of boundary conditions becomes necessary as well as the restriction of allowed classes of wave functions. The aim of this article is to study some quantum mechanical theorems in polar spherical coordinates when the area is not a full space. We will see that in most cases great caution must be exercised especially in cases when the potential of the Schrodinger equation is singular or the operator is singular itself.

2. Time derivative of mean values of operators

It is well known that in quantum mechanics derivative of time-dependent operator $\hat{A}(t)$ is transferred from the corresponding classical expression according to replacement of the Poison bracket by quantum commutator [5, 6]:

$$\frac{d\hat{A}}{dt} = \frac{\partial \hat{A}}{\partial t} + \frac{i}{\hbar} \left[ \hat{H}, \hat{A} \right].$$

If one averages this expression by state function, it follows

$$\left\langle \frac{d\hat{A}}{dt} \right\rangle = \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle + \frac{i}{\hbar} \left\langle \left[ \hat{H}, \hat{A} \right] \right\rangle. \quad (1)$$

As a rule one believes that these two operations – time derivative and average procedures can be interchanged [2]. This is postulated as a definition.
\[
\langle \frac{d\hat{A}}{dt} \rangle = \frac{d}{dt} \langle \hat{A} \rangle = \frac{\partial \hat{A}}{\partial t} + \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle.
\]

(2)

Let us see, is it so or not in general? By this aim we calculate the derivative

\[
\frac{d}{dt} \langle \hat{A} \rangle = \frac{d}{dt} \langle \psi | \hat{A} | \psi \rangle = \langle \psi | \frac{\partial \hat{A}}{\partial t} | \psi \rangle + \langle \psi | \hat{A} \frac{\partial \psi}{\partial t} | \psi \rangle + \langle \psi | \hat{A} \frac{\partial \psi}{\partial t} | \psi \rangle.
\]

(3)

Assume that the operator \(\hat{A}\) has a central symmetry, \(\hat{A} = \hat{A}(r, \ldots)\) and study terms entering equation (3).

The time dependent Schrodinger equation in the first and third terms of equation (3)

\[
i \hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi \quad \text{and} \quad -i \hbar \frac{\partial \psi^*}{\partial t} = (\hat{H})^* \psi
\]

(4)

and take the Hamiltonian in the radial form

\[
H = \frac{1}{2m} \left(-\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} \right) + \frac{l(l+1)}{2mr^2} + U(r, t).
\]

(5)

Remark that the potential energy must depend on time, otherwise wave functions be stationary. Using the Schrodinger equations (4) in (3) with the Hamiltonian (5) and performing two-fold partial integration in the derivative terms of integrand, in order to permute differentiation to the right and organize the radial Hamiltonian again, we obtain at the end

\[
\frac{d}{dt} \langle \hat{A} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle + \langle \frac{\partial \hat{A}}{\partial t} \rangle + \Pi.
\]

(6)

Where the additional term appears

\[
\Pi = \frac{i}{\hbar} \frac{1}{2m} \lim_{r \to 0} \left\{ r^2 \left[ \hat{A} R \frac{dR^*}{dr} - R^* \frac{d}{dr}(\hat{A} R) \right] \right\}.
\]

(7)

This term is not zero in general, because it depends on the behavior of wave function and operator at the origin of coordinates.

Under rather general requirements, the full radial function must be behaved as [3,4]

\[
r R(r) = 0.
\]

(8)

More particularly the behavior of radial wave function depends on the potential under consideration.

Let us first consider regular potentials, when \(R \sim C_1 r^l\). It is obvious from equation (8) that upon calculation of the limit the singularity of an operator \(\hat{A}\) at the origin will be also important. We take it as

\[
\hat{A}(r) \sim 1/r^\beta; \quad (\beta > 0).
\]

(9)

Then we have

\[
\Pi = \frac{i}{\hbar} \frac{C_1^2}{2m} \lim_{r \to 0} r^2 \left\{ r^{l-\beta} r^{l-1} - r^l \frac{d}{dr} r^{l-\beta} \right\} = \frac{i}{\hbar} \frac{C_1^2}{2m} \lim_{r \to 0} r^{2l+1-\beta}.
\]

(10)

In order this expression will not be diverging we must require

\[
2l + 1 > \beta.
\]

(11)
In this case the additional term vanishes. If the inequality be reflected then the divergent result follows and we were unable to write the equation (1).

On the other hand, if the operator is such that

\[ 2l + 1 = \beta, \] (12)

then the extra term remains on the right-hand side

\[ \frac{d}{dt} \langle \hat{A} \rangle = \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle + i \frac{\hbar}{\hbar} \left\langle \left[ \hat{H}, \hat{A} \right] \right\rangle + iC^2 \left( \frac{1}{m} (\beta + 1/2) \right). \]

We see that the averaging relation is not so trivial, as it looks at the first glance, but is depending on singularity of operator under consideration.

We can conclude that even in ordinary quantum mechanics the well-known averaging relation is validate only in cases, when the condition (12) between orbital momentum and operator singularity at the origin is satisfied. At the same time for "soft" singular potential [3,4] the standard solutions some analogous restrictions. It is evident that this strange result is provided by singular character of considered operator. Particularly the strangest is the fact that the time derivative of average value does not coincide to the average of derivative of the operator.

In conclusion, we see that if the operator has a "bad" singularity, its average value is not an integral of motion if it even commutes with the Hamiltonian. Moreover, many famous theorems like Ehrenfest one [1] as well as lot of hypervirial relations [2] may be modified.

3. Acknowledgments

This work was supported by Shota Rustaveli National Science Foundation (SRNSF) [grant number No DI-2016-26, Project Title: “Three-particle problem in a box and in the continuum”]

References

PRINCIPAL FUNCTIONS OF DISCRETE STURM-LIOUVILLE EQUATION WITH GENERALIZED EIGENPARAMETER IN BOUNDARY CONDITION

TURHAN KOPRUBASI

1Kastamonu University, Department of Mathematics, Turkey
e-mail: tkoprubasi@kastamonu.edu.tr

Spectral analysis of difference equations has been intensively studied since 1960. The modeling to problems of certain areas such as engineering, biology, economics, control theory and other areas of study has led to the rapid development of the theory of difference equations. The spectral theory of discrete equations has also been applied to the solution of classes of nonlinear discrete equations and Toda lattices [19]. Moreover, some problems of spectral theory of difference equations relevant to the classical moment problem were studied by various authors ([3, 5, 9-12, 18]).

Let us consider the boundary value problem (BVP)

\[ \begin{align*}
- u'' + q(x)u &= \lambda^2 u, & 0 \leq x < \infty \\
 u'(0) - hu(0) &= 0,
\end{align*} \]

(1)

where \( h \in \mathbb{C} \), \( q \) is a complex valued function, and \( \lambda \) is a spectral parameter.

The spectral properties of the BVP (1) has been investigated in [17] and it is concluded that the spectral singularities are poles of the resolvent that are imbedded in the continuous spectrum and are not eigenvalues. The role of spectral singularities bearing on the spectral expansion has been discussed in [16]. The discrete spectrum of general difference equations has been studied in [4]. Related to the second order equations studied in this paper are the results of [2] where Adivar has studied the quadratic pencil of difference equations. The spectral analysis of a non-selfadjoint second order difference equation with principal functions has been investigated in [1]. In that study, it is shown that the Jost solution of this equation has an analytic continuation to the lower half-plane and the finiteness of the eigenvalues and the spectral singularities of the difference equation is obtained as a result of this analytic continuation. Some problems about spectral theory of difference equations with spectral singularities have been debated in [7, 15]. The spectral analysis of non-selfadjoint difference equation and Sturm Liouville equation problem which is eigenparameter dependent was studied in [6, 8, 13].

Let the boundary value problem,

\[ \begin{align*}
 a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} &= \lambda y_n, & n \in \mathbb{N} \\
 \sum_{k=0}^{p} (y_1 \gamma_k + y_0 \beta_k) \lambda^k &= 0
\end{align*} \]

(2)

is considered where \( \lambda \) is a spectral parameter, \( \{a_n\}_{n\in\mathbb{N}} \) and \( \{b_n\}_{n\in\mathbb{N}} \) are complex sequences, \( a_n \neq 0 \) for all \( n \in \mathbb{N} \cup \{0\} \), \( \gamma_0 \beta_1 - \gamma_1 \beta_0 \neq 0 \), \( \sum_{k=0}^{p} |\gamma_k| + |\beta_k| \neq 0 \) and \( \gamma_p \neq -\frac{\beta_{p-1}}{a_0} \), where \( \gamma_k, \beta_k \in \mathbb{C}, k = 0, 1, 2, \ldots, p \). In here, we can write the difference equation (2) in the following Sturm-Liouville form:
\[ \Delta(a_{n-1} \Delta y_{n-1}) + h_n y_n = \lambda y_n, \quad n \in \mathbb{N}, \]

where \( \Delta \) forward difference operator with
\[ h_n = a_{n-1} + a_n + b_n. \]

In this study, several spectral properties of principal functions of the BVP (2)-(3) described by
\[
V^{(\xi)}(\lambda_j) = \frac{1}{\xi!} \left. \left\{ \frac{d^\xi}{d\lambda^\xi} E_n(\lambda) \right\} \right|_{\lambda=\lambda_j},
\]
\[ \xi = 0, 1, \ldots, m_j - 1 ; \quad j = 1, 2, \ldots, r, \]

\[
V^{(\xi)}(\lambda_j) = \frac{1}{\xi!} \left. \left\{ \frac{d^\xi}{d\lambda^\xi} E_n(\lambda) \right\} \right|_{\lambda=\lambda_j},
\]
\[ \xi = 0, 1, \ldots, m_j - 1 ; \quad j = r + 1, r + 2, \ldots, q \]

with
\[
\{ E_n(\lambda) \} := \left\{ e_n(\arccos \frac{\lambda}{2}) \right\}, \quad n \in \mathbb{N}
\]

are mentioned under the condition
\[
\sum_{n=1}^{\infty} n (|1 - a_n| + |b_n|) < \infty,
\]
where \( \lambda_1, \lambda_2, \ldots, \lambda_r \) and \( \lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_q \) denote the zeros of Jost function
\[ e_n(z) = \alpha_n e^{inz} \left( 1 + \sum_{m=1}^{\infty} A_{nm} e^{imz} \right), \quad n \in \mathbb{N} \cup \{0\}, \]
in
\[ P_0 := \{ z : z \in \mathbb{C}, \quad z = x + iy, \quad -\pi \leq x \leq \pi, \quad y > 0 \}
\]
and its lower boundary \([-\pi, \pi]\) with multiplicities \( m_1, m_2, \ldots, m_r \) and \( m_{r+1}, m_{r+2}, \ldots, m_q \), respectively ([14]). In here, \( z \in \mathbb{C}_+ := \{ z : z \in \mathbb{C}, \quad Imz \geq 0 \} \), and \( \alpha_n, A_{nm} \) are expressed in terms of \( \{a_n\}_{n \in \mathbb{N}} \) and \( \{b_n\}_{n \in \mathbb{N}} \) as
\[
\alpha_n = \left( \prod_{\xi=n}^{\infty} a_\xi \right)^{-1},
\]
\[
A_{n,1} = - \sum_{\xi=n+1}^{\infty} b_\xi,
\]
\[
A_{n,2} = - \sum_{\xi=n+1}^{\infty} (1 - a_\xi^2) + \sum_{\xi=n+1}^{\infty} b_\xi \sum_{p=\xi+1}^{\infty} \sum_{p=\xi+1}^{\infty} b_p,
\]
\[
A_{n,m+2} = \sum_{\xi=n+1}^{\infty} (1 - a_\xi^2) A_{\xi+1,m} \sum_{\xi=n+1}^{\infty} b_\xi A_{\xi,m+1} + A_{n+1,m}.
\]

Differently other studies in the literature, the specific feature of this work which is one of the studies have applicability in study areas such as control theory, economics, medicine and biology is the presence of the spectral parameter not only in the difference equation but also it’s in the boundary condition at generalized polynomial form.
Keywords: Discrete equations, Eigenparameter, spectral analysis, eigenvalues, spectral singularities, principal functions.

AMS Subject Classification: 34L05, 34L40, 39A70, 47A10, 47A75.

References


PECULIARITIES OF FORMING THE PIEZOELECTRIC EFFECT IN THE COMPOSITE OF POLYMER-SEGNETOPEOSERAMICSS

M.A. KURBANOV¹, B.H. KHUDAYAROV¹, I.S. RAMAZANOVA¹, G.KH. HUSEYNOVA¹, A.F. NURALIEV¹, Z.A. DADASHOV¹, O.A. ALIYEV¹

¹Institute of Physics, National Academy of Sciences of Azerbaijan, Baku, Azerbaijan
e-mail: mKurbanov@physics.ab.az

Abstract. The paper considers the features of the formation of the piezoelectric effect in polymers dispersed by ferroelectric piezoelectrics of various structures. It has been established that piezocomposites based on polymer and ferroelectric ceramics with a rhombohedral structure have high piezoelectric coefficients, some of them exhibiting anomalously high values of $d_{ij}$ and piezosensitivity $g_{ij}$ under certain polarization conditions.

Keywords: Triple systems, automotive, oil, nuclear, maritime, aviation sectors.

AMS Subject Classification: 74E05, 74E30, 74H10, 74H55.

1. Introduction

At present, there are basically three directions in the field of creating active dielectric elements based on solid materials. One of them is related to the creation of newer and newer ferroelectric materials based on barium titanate and lead zirconate titanate. The second is based on the synthesis of new polymeric dielectrics that exhibit electret, pyro-and piezoelectric properties [1-5]. The new opportunity opens up a third direction, related to the creation of various composite active materials based on polymers (matrix) dispersed by inorganic ferroelectric piezoelectric fillers.

2. Experimental method

The piezoelectric module when compressing the piezoelement is defined as $d_{33} = U_k S_{co}/\sigma S$ where $U_k$ is the voltage arising on the piezoelectric element; $S_{co} = S_0 + S_{wi}$ -total capacitance of the measuring system; $\sigma$-mechanical stress; $S$-area of the measuring electrode, $S_{wi}$ -capacity of the connecting wires, $S_0$-capacitance, which is parallel to the piezoelectric element.

3. Experimental results and their discussion

For the analysis of polarization phenomena in composites, the temperature dependences of their piezomodule $d_{33}$ and the electret potential difference $U_p$ were investigated. It can be seen that the nature of the variation in the temperature dependences (Fig.1) of $d_{33}$ for composites both on the basis of polar and nonpolar polymers practically coincide [5-7]. The piezoelectric and specific properties of composites based on polyvinylidene fluoride are more stable depending on the temperature, for example, $d_{33}$ of the PVDF + PKR-3M composite practically does not change to approximately 433 K (Fig.1 a). The piezoelectric characteristics of the LDPE + PKR-3M composite are less stable (Fig.1 b). From the results shown in Fig.1 for the specific
potential difference $U_p$ of the composites, it is seen that the temperature at which the decay $U_p$ is observed is noticeably smaller than the value of the decay temperature $d_{33}$. For example, for the PVDF + PKR-3M composite, the decrease in the electret potential difference begins at temperatures slightly above room temperature, and the decrease in $d_{33}$ from the temperature begins at temperatures above 353 K. This fact shows that in the process of thermoelectricity of the piezocomposite, charges form in its volume, some of which do not participate in the piezoelectric response. The low temperature of their decay indicates that these charges are mainly stabilized on shallow traps. These results show that the constancy of the dipole polarization and the stabilization of the injected charges are to a certain extent determined by the properties of the matrix itself.

Under the conditions of our experiments for the piezocomposites studied, the sign of the electret potential difference corresponded to the sign of the homo charge and indicates that part of the injected charges was compensated by oriented domains. The piezoparticles form a fairly stable system, and some of them determine the value of $U_p$. Indeed, the piezomodule is proportional to the product of the reorientational polarization $P_r$ by the dielectric constant, i.e,

$$d_{33} = P_r \varepsilon_{33}.$$ 

As already noted, the reorientational polarization $P_r$ in the transition from the rhombohedral to the tetragonal region decreases, while the permittivity $\varepsilon_{33}$ increases and reaches a maximum in the $T$ region. Therefore, although the $P_r$ piezoceramic from the PE region is larger than that of the ceramics from the $T$ region, the $\varepsilon_{33}$ of the former are considerably larger than the ones of the latter. For example, in PKR-3M, $\varepsilon_{33} = 400$ and $P_r = 0.866P_s$, and for PKR-7M, $\varepsilon_{33} = 7600$ and $P_r = 0.831P_s$, therefore, $d_{33}$ of piezoceramics of PKR-3M is proportional to $346.6P_s$, and for PKR-7M to $6315.6P_s$.

![Figure 1. A) $PVDF + PKR - 3M. = 50\%$, $p = 393,1, 2, 3 - d_{33} = f(T_{33})$ respectively polarized at $p = 1, 5 : 3, 0 : 4, 5mV/m, 4, 5, 6 - U_p = f(T_{us})$, respectively, which are polarized at $E_n = 1.5 : 3.0 : 4.5mV/m$.](image)

However, the dielectric constant of polymer piezocomposites with various ferroelectric piezoceramics does not differ much from each other. Therefore, the piezoelectric modulus of the composites is mainly determined by the reorientational polarization $P_r$ in the piezoceramic particles, which is higher in the rhombohedral phase ceramics.
Figure 2. B) LDPE + PKR-3M.

242

M.A. KURBANOV, ET AL

4. CONCLUSION

Measurements of the thermally stimulated depolarization current and the temperature dependence of the electret potential difference are used to determine the interphase interactions in composites, depending on the structure and electronegativity of the piezophase cations and macromolecules of the polymer matrix.

REFERENCES

OPTIMIZATION OF MAYER PROBLEM WITH DIFFERENTIAL INCLUSIONS AND POLYNOMIAL DIFFERENTIAL OPERATORS

ELIMHAN N. MAHMUDOV

1Department of Mathematics, Istanbul Technical University Istanbul, Turkey
2Azerbaijan National Academy of Sciences Institute of Control Systems Baku, Azerbaijan
e-mail: elimhan22@yahoo.com

The present paper studies a new class of problems of optimal control theory with differential inclusions described by polynomial linear differential operators (PLDOs). Consequently, there arises a rather complicated problem with simultaneous determination of the PLDOs with variable coefficients and a Mayer functional depending of high order derivatives of searched functions. The sufficient conditions, containing both the Euler-Lagrange and Hamiltonian type inclusions and "transversality" conditions are derived. Formulation of the transversality conditions at the endpoints \( t = 0 \) and \( t = 1 \) of the considered time interval play a substantial role in the next investigations without which it is hardly ever possible to get any optimality conditions. The main idea of the proof of optimality conditions of Mayer problem for differential inclusions with PLDO is the use of locally-adjoint mappings [1]-[5]. In general, optimization of higher order differential inclusions arise from a wide variety of problems in science and engineering. In each case, their successful resolution requires the specific peculiarities of each problem to be included in the mathematical model. In our presentation, we discuss a special kind of optimization problem with differential inclusions, in which the left hand side of the inclusion is PLDOs with variable coefficients. In particular, it is shown that our problem involve optimization of so-called Sturm-Liouville type differential inclusions. To the best of our knowledge, there is no paper which considers optimality conditions for these problems in the literature and we aim to fill this gap. Therefore, the novelty of our formulation of the problem is justified. Notice that the proof relies on consideration of a convex case, even though the result remains true for nonconvex problem, too. Furthermore, practical applications of these results are demonstrated by optimization of some semilinear with respect to the state variable optimal control problems for which the Weierstrass-Pontryagin maximum condition [7] is obtained. We recall the key notions of set-valued mappings from the book [1]; let \( R^n \) be a \( n \)-dimensional Euclidean space, \((x, v)\) be an inner product of elements \( x, v \in R^n \), \((x, v)\) be a pair of \( x, v \). Let \( F : R^n \rightrightarrows R^n \) be a set-valued mapping from \( R^n \) into the set of subsets of \( R^n \). The Hamiltonian function and argmaximum set corresponding to a set-valued mapping \( F \) are denoted by \( H_F(x, v^*) \) and \( F_A(x, v^*) \), respectively. A set-valued mapping \( F^* : R^n \rightrightarrows R^n \) defined by \( F^*(v^*, (x, v)) = \{x^* : (x^*, -v^*) \in K_{gphF}(x, v)\} \) is called the LAM to \( F \) at a point \((x, v)\) \( \in gphF \), where \( K^* \) denotes the dual cone to the cone \( K \), as usual. The LAM to "nonconvex" mapping \( F \) is defined as follows \( F^*(v^*, (x, v)) := \{x^* : H_F(x^1, v^*) - H_F(x, v^*) \leq \langle x^*, x^1 - x \rangle, \forall x^1 \in R^n \}, (x, v) \in gphF \). Note that prior to the LAM the notion of coderivative has been introduced for set-valued mappings in terms of the basic normal cone to their graphs by Mordukhovich [6]. Our goal is to give Euler-Lagrange and Hamiltonian optimality conditions for the following general Mayer problem governed by ordinary differential inclusions with PLDOs and with initial point constraints.
(PV) minimize $\varphi(x(1), x'(1), ..., x^{(m-1)}(1))$

subject to

$$Lx(t) \in F(x(t), t),$$

$$x(0) \in Q_0, x'(0) \in Q_1, x''(0) \in Q_2, ..., x^{(m-1)}(0) \in Q_{m-1}, \text{ a.e. } t \in [0, 1],$$

where $Lx = \sum_{k=1}^{m} p_k(t) D^k x$ is a PLDO of degree $m$ with variable coefficients $p_k : [0, 1] \to \mathbb{R}^1$ and $(m \geq 2)$ $D^k, k = 1, ..., m$ is the operator of $k$-th order derivatives. In what follows for each $k$ a scalar function $p_k$ is $k$-th order continuously differentiable function, $p_m(t) \neq 0$ on $[0, 1]$ identically, $F(\cdot, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is time dependent set-valued mapping, $\varphi : (\mathbb{R}^n)^m \to \mathbb{R}^1$ is continuous function and $Q_j \subseteq \mathbb{R}^n, j = 0, 1, ..., m - 1$ are nonempty subsets of $\mathbb{R}^n$. We label this problem as $(PV)$. First of all we associate with the problem $(PV)$ the following so-called the $m$-th order Euler-Lagrange differential inclusion with PLDO and the transversality conditions at the endpoints $t = 0$ and $t = 1$:

(i) $L^* x^*(t) \in F^*(x^*(t); (\tilde{x}(t), L\tilde{x}(t)), t)$, a.e. $t \in [0, 1]$, where

$$L^* x^*(t) = \sum_{k=1}^{m} (-1)^k D^k [p_k(t)x^*(t)]$$

is the adjoint PLDO operator,

(ii) $\sum_{k=0}^{m-1} (-1)^{m-k} D^{m-k-1} [p_{m-k}(0)x^*(0)] \in K^*_{q_0}(\tilde{x}(0))$

$$\sum_{k=0}^{m-2} (-1)^{m-k-1} D^{m-k-2} [p_{m-k}(0)x^*(0)]$$

$$\in K^*_{q_1}(\tilde{x}(0)); \cdots ; D[p_m(0)x^*(0)] - p_{m-1}(0)x^*(0) \in K^*_{q_{m-2}}(\tilde{x}^{(m-2)}(0));$$

$$-p_m(0)x^*(0) \in K_{q_{m-1}}^*(\tilde{x}^{(m-1)}(0)).$$

(iii) $\left(\sum_{k=0}^{m-1} (-1)^{m-k} D^{m-k-1} [p_{m-k}(1)x^*(1)],$

$$\sum_{k=0}^{m-2} (-1)^{m-k-1} D^{m-k-2} [p_{m-k}(1)x^*(1)],$$

$$\cdots , D[p_m(1)x^*(1)] - p_{m-1}(1)x^*(1), -p_m(1)x^*(1)\right)$

$$\in \partial \varphi(\tilde{x}(1), \tilde{x}'(1), ..., \tilde{x}^{(m-1)}(1)).$$

At last we formulate the condition ensuring that the LAM $F^*$ is nonempty at a given point:

(iv) $L\tilde{x}(t) \in F_A(\tilde{x}(t), x^*(t), t)$, a.e. $t \in [0, 1]$. Under the conditions (i)-(iv) are derived sufficient conditions of optimality for a problem $(PV)$.

**Example 1.** We consider the following Mayer problem with second order PLDO $Lx = D^2 x = x''$, the value of which is $-7/10$:

minimize $\varphi(x(1), x'(1))$
subject to
\[ x'' = u, \quad u \in [-1, 1], \quad x(0) \in Q_0, \quad x'(0) \in Q_1, \]
where
\[ \varphi(x(1), x'(1)) = x'^2(1) - x(1) \] and \( Q_0 = \{0\}, Q_1 = \{1\}. \]

**Keywords:** Euler-Lagrange, set-valued, polynomial operators, transversality.

**AMS Subject Classification:** 34A60, 49J15, 49K15, 65J10.

**References**

FINITE-APPROXIMATE CONTROLLABILITY OF SEMILINEAR EVOLUTION SYSTEMS VIA RESOLVENT-LIKE OPERATORS

N.I. MAHMUDOV

1Department of Mathematics, Eastern Mediterranean University, Famagusta, T.R. North Cyprus, Turkey
e-mail: nazim.mahmudov@emu.edu.tr

Controllability is one of the basic qualitative concepts in modern mathematical control theory that play an important role in deterministic and stochastic control theory. From mathematical point of view, exact and approximate controllability problems should be distinguished. Exact controllability enables to steer the system to an arbitrary final state while approximate controllability means that the system can be steered to an arbitrary small neighborhood of final state, and very often approximate controllability is completely adequate in applications. If the semigroup associated with the system is compact, the controllability operator is also compact, and therefore the inverse fails to exist. Hence, the concept of exact controllability is very strong and feasibility is limited; approximately controllability is a weaker concept that is entirely appropriate in practice. There are many papers on the approximate controllability of the various types of nonlinear systems under different conditions (see [2, 11] and references therein).

In this paper we will study a stronger version of controllability concept that is referred to as the finite-approximate controllability problem. It should be stressed out that in the context of abstract linear control systems, finite-approximate controllability problem is a consequence of approximate one, see [8]. So these two concepts are equivalent. However in the nonlinear context they are not equivalent, see [12]. Recently finite-approximate controllability result for abstract semilinear evolution equations with compact $C_0$-semigroup is presented in [9].

In this paper, we investigate simultaneous approximate and finite-dimensional exact controllability (finite-approximate controllability) of the following semilinear evolution system:

\[
\begin{align*}
    y'(t) &= Ay(t) + Bu(t) + f(t, y(t)) + g(t, y(t)), \quad t \in [0, T], \\
    y(0) &= y_0,
\end{align*}
\]

(1)

where the state variable $y(\cdot)$ takes values in the Hilbert space $X$, $A : D(A) \subset X \to X$ is a family of closed and bounded linear operators generating a strongly continuous semigroup $\mathcal{U} : [0, T] \to L(X)$, where the domain $D(A) \subset X$ which is dense in $X$, the control function $u(\cdot)$ is given in $L^2([0, T], U)$, $U$ is a Hilbert space, $B$ is a bounded linear operator from $U$ into $X$, $f, g : [0, T] \times X \to X$ are given functions satisfying some assumptions specified later and $y_0$ is an element of the Hilbert space $X$.

We present the following definition of mild solutions of system (1).

**Definition 1.** $y \in C([0, T], X)$ is called a mild solution of (1) if

\[
y(t) = \mathcal{U}(t)x_0 + \int_0^t \mathcal{U}(t-s)[Bu(s) + f(s, y(s)) + g(s, y(s))]ds, \quad t \in [0, T].
\]

(2)

Following [1], we define the controllability concepts and controllability operator for the system (1).
Definition 2. For the system (1), we define the following concepts:
(a) Control system (1) is approximately controllable on \([0, T]\), if for every \(y_0, y_f \in \mathcal{X}\), and for every \(\varepsilon > 0\), there exists a control \(u \in L^2([0, T], U)\) such that the mild solution \(y\) of the Cauchy problem (1) satisfies \(y(0) = y_0\) and \(\|y(T) - y_f\| < \varepsilon\).
(b) Let \(M\) be a finite dimensional subspace of \(\mathcal{X}\) and let us denote by \(\pi_M\) the orthogonal projection from \(\mathcal{X}\) into \(M\). Control system (1) is finite-approximately controllable on \([0, T]\), if for every \(y_0, y_f \in \mathcal{X}\), and for every \(\varepsilon > 0\), there exists a control \(u \in L^2([0, T], U)\) such that the mild solution \(y\) of the Cauchy problem (1) satisfies \(y(0) = y_0\) and \(\|y(T) - y_f\| < \varepsilon\) and \(\pi_M y(T) = \pi_M y_f\).

Firstly, we investigate the finite-approximate controllability of linear evolution system:
\[
\begin{align*}
\frac{dy}{dt} &= Ay(t) + Bu(t), & t &\in [0, T], \\
y(0) &= y_0.
\end{align*}
\] (3)

Finite-approximate controllability concept was introduced in [12]. This property not only says that the distance between \(y(T)\) and the target \(y_f\) is small but also that the projections of \(y(T)\) and \(y_f\) over \(M\) coincide.

It is known that the resolvent operator \((\varepsilon I + \Gamma^0_\varepsilon)^{-1}\) is useful in studying the controllability properties of linear and semilinear systems, see [2, 9]. In this respect, we state a useful characterization of the finite-approximate controllability for (3) in terms of resolvent-like operator. We show that for the linear evolution system (3) approximate controllability on \([0, T]\) is equivalent to the finite-approximate controllability on \([0, T]\). Moreover, we present necessary and sufficient conditions for the finite-approximate controllability of linear evolution systems in Hilbert spaces in terms of resolvent-like operators.

Firstly, we present three results on the resolvent operators.

**Theorem 1.** Assume that \(\Gamma(\varepsilon), \Gamma : \mathcal{X} \to \mathcal{X}, \varepsilon > 0\), are linear positive operators such that
\[
\lim_{\varepsilon \to 0^+} \|\Gamma(\varepsilon)h - \Gamma h\| = 0, \ h \in \mathcal{X}.
\]

Then for any sequence \(\{\varepsilon_n > 0\}\) converging to \(0\) as \(n \to \infty\), we have
\[
\lim_{n \to \infty} \varepsilon_n (\varepsilon_n I + \Gamma(\varepsilon_n))^{-1} \pi_M = 0.
\]

**Theorem 2.** Assume that \(\Gamma(\varepsilon) : \mathcal{X} \to \mathcal{X}, \varepsilon > 0\), are linear positive operators. Then for any \(\varepsilon > 0\) we have \(\|\varepsilon (\varepsilon I + \Gamma(\varepsilon))^{-1} \pi_M\| < 1\).

**Theorem 3.** If \(\Gamma : \mathcal{X} \to \mathcal{X}\) is a linear nonnegative operator then the operator \(\varepsilon (I - \pi_M) + \Gamma : \mathcal{X} \to \mathcal{X}\) is invertible and
\[
\left\|(\varepsilon (I - \pi_M) + \Gamma)^{-1} h\right\| \leq \frac{1}{\min(\varepsilon, \delta)} \|h\|, \ h \in \mathcal{X},
\] (4)

where \(\delta = \min\{\pi_M \Gamma \pi_M \varphi, \varphi : \|\pi_M \varphi\| = 1\}\). Moreover, if \(\Gamma : \mathcal{X} \to \mathcal{X}\) is a linear positive operator then
\[
(\varepsilon (I - \pi_M) + \Gamma)^{-1} = \left(I - \varepsilon (\varepsilon I + \Gamma)^{-1} \pi_M\right)^{-1} (\varepsilon I + \Gamma)^{-1}.
\] (5)

Next, we present new criteria for the finite-approximate controllability of linear evolution equations.

**Theorem 4.** The following statements are equivalent:
(i) the system (3) is approximately controllable on \([0, T]\);
(ii) \(\Gamma^0_0\) is positive, that is \(\langle \Gamma^0_0 x, x \rangle > 0\) for all \(0 \neq x \in \mathcal{X}\);
(iii) \(\varepsilon (\varepsilon I + \Gamma^0_\varepsilon)^{-1} \to 0\) as \(\varepsilon \to 0^+\) in the strong operator topology;
(iv) \(\varepsilon (\varepsilon (I - \pi_M) + \Gamma^0_\varepsilon)^{-1} \to 0\) as \(\varepsilon \to 0^+\) in the strong operator topology;
(v) the system (3) is finite-approximately controllable on \([0, T]\).
Analogue of Theorem 4 is true for different kind of equations such as fractional linear differential equations with Caputo derivative, fractional linear differential equations with Riemann-Liouville derivative, Fredholm type linear integral equations and so on.

We impose the following assumptions:

(S) \( X \) and \( U \) are separable Hilbert spaces, \( \Upsilon(t), t > 0 \) is a compact semigroup on \( X \) and \( B \in L(U, X) \).

(F) \( f : [0, T] \times X \rightarrow X \) is continuous and has continuous uniformly bounded Fréchet derivative \( f'_z (\cdot, \cdot) \), that is, for some \( L > 0 \),
\[
\| f'_z (t, z) \|_{L(X)} \leq L, \quad \forall (t, z) \in [0, T] \times X.
\]

(G) \( g : [0, T] \times X \rightarrow X \) is continuous and there exists \( m \in C([0, T], R^+) \) such that
\[
\| g(t, z) \| \leq m(t), \quad \forall (t, z) \in [0, T] \times X.
\]

(AC) System
\[
y(t) = \Upsilon(t)y_0 + \int_0^t \Upsilon(t-s) [Bu(s) + G(s)y(s)] ds
\]
(6)
is approximately controllable for any \( G \in L^2(0, T; L(X)) \).

**Theorem 5.** Under the conditions (S), (F), (G), (AC) the system (1) is finite-approximately controllable on \([0, T]\).

**Keywords:** Approximate controllability, finite dimensional exact controllability, semigroup, evolution equation.

**AMS Subject Classification:** 34G20, 93B05.

**References**


3D OPTIMAL CONTROL PROBLEM FOR A MANJERON GENERALIZED EQUATION WITH NON-CLASSICAL GOURSAT CONDITIONS

I.G. MAMEDOV

Institute of Control Systems of NAS of Azerbaijan, Baku, Azerbaijan
e-mail: ilgar-mammadov@rambler.ru

In this paper we consider a 3D (three-dimensional) optimal control problem with distributed parameters for the 3D Manjeron generalized equation with nonsmooth coefficients of the sixth order.

1. INTRODUCTION

It is well known that various optimal control problems described by hyperbolic equation, as well as the equations of mathematical physics at various assumptions obtained some necessary and sufficient conditions of optimality. Development of optimal control theory led to its application to practical problems, such as a controlled object, optimization of dynamical systems and others. Many of these optimal control problems, the solution of which is the subject of numerous works, described by hyperbolic equations. The problem of optimal control of systems with distributed parameters has numerous applications.

In this paper the 3D optimal control problem for a Manjeron generalized equation with nonsmooth coefficients with nonclassical Goursat boundary value problems is investigated. The statement of 3D optimal control problem is studied by using a new version of the increment method that essentially uses the concept of the adjoint equation of the integral form. The method also includes the case where the coefficients of the equation are nonsmooth functions. In the paper it is shown that such an 3D optimal control problem can be investigated with the help of a new concept of the adjoint equation, which can be regarded as an auxiliary equation for determination of Lagrange multipliers.

2. PROBLEM STATEMENT

Let the controlled object be described by the 3D (three-dimensional) Manjeron generalized equation:

\[ (V_{2,2,2} u)(x) \equiv D_1^2 D_2^2 D_3^2 u(x) + \sum_{i_1=0}^2 \sum_{i_2=0}^2 \sum_{i_3=0}^2 a_{i_1i_2i_3}(x) D_1^{i_1} D_2^{i_2} D_3^{i_3} u(x) = \varphi(x, v(x)), \quad (1) \]

\[ x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad G_k = (x_k^0, h_k), \quad k = 1, 3; \]
under the following non-classical 3D Goursat conditions:

\[
\begin{align*}
(V_{1,1,2,3} u)(x_1) &\equiv D^1_0 D^2_1 D^3_3 u(x)/x_3 = Z_{1,1,2,3}(x_1) \in L_p(G_1), \quad i_k = 0, 1, k = 1, 3; \\
(V_{2,1,2,3} u)(x_2) &\equiv D^1_0 D^2_1 D^3_3 u(x)/x_3 = Z_{2,1,2,3}(x_2) \in L_p(G_2), \quad i_k = 0, 1, k = 1, 3; \\
(V_{1,1,2,2} u)(x_3) &\equiv D^1_0 D^2_1 D^3_3 u(x)/x_3 = Z_{1,1,2,2}(x_3) \in L_p(G_3), \quad i_k = 0, 1, i_2 = 0, 1, \\
(V_{2,1,2,2} u)(x_4) &\equiv D^1_0 D^2_1 D^3_3 u(x)/x_3 = Z_{2,1,2,2}(x_4) \in L_p(G_4), \quad i_k = 0, 1, i_2 = 0, 1, \quad (2)
\end{align*}
\]

where \( D^i_k = \partial^i / \partial x^i \), \( k = 1, 3 \) is generalized differentiation operator in the S.L.Sobolev sense; \( u(x) \) is the desired function; \( a_{i_1,i_2,i_3}(x) \) are the given measurable functions on \( G \). \( \varphi(x, v(x)) \) are the given functions on \( G \times R^r \) satisfying the Caratheodory conditions; \( v(x) = (v_1(x), ..., v_r(x)) \) is \( r \)-dimensional controlling vector-function; \( Z_{i_1,i_2,i_3} \) are the given elements.

Let the function \( v(x) \) measurable and bounded on \( G \) and almost at all points of \( x \in G \) accept its values from some given set \( \Omega \subseteq R^r \). Then this vector-function is said to be an admissible control. A set of all admissible controls is denoted by \( \Omega \).

In the paper we consider the following linear problem of optimal control: find admissible control \( v(x) \) from \( \Omega \), for which the solution

\[
\begin{array}{l}
u(x) \in W^{2,2,2}_p(G) \equiv \{ u(x) : D^1_0 D^2_1 D^3_3 u(x) \in L_p(G), \quad i_k = 0, 1, 2, k = 1, 2, 3 \};
\end{array}
\]

(1 \( \leq p \leq \infty \)) of 3D Goursat boundary-problem (1)-(2) delivers the least value to the 3D linear multi-point functional:

\[
S(v) = \sum_{k=1}^{N} b_k \left[ D_1 u(x^k) + D_2 u(x^k) + D_3 u(x^k) \right] +
\]

\[
+ c_k \left[ D_1 D_2 u(x^k) + D_2 D_3 u(x^k) + D_1 D_3 u(x^k) \right] \to \min,
\]

(3)

where \( x^k = (x^k_1, x^k_2, x^k_3) \) are the given points; \( b_k, c_k \in R \) are the given numbers. Equation (1) is a hyperbolic equation with three double real characteristics \( x_k = \text{const}, \quad k = 1, 3 \). Therefore, in some sense we can consider equation (1) as a pseudoparabolic equation. This equation is a Boussinesq - Love generalized equation from the vibrations theory [9] and Aller’s equation under mathematical modeling [7, p.261] of the moisture absorption process in biology.

The necessary and sufficient conditions for optimal process for ordinary differential equations and partial differential equations under local conditions sufficiently thoroughly studied by many mathematicians. The results obtained in this area studied in detail in monographs such as [5], [8], [10] and others. Fundamental solution of certain local and non-local boundary value problems [2] and its applications to optimization are investigated in work [3].

The 3D optimal control problem (1)-(3) was studied by means of a new variant the increment method. The method essentially uses the notion of an integral form conjugation equation and allows also to cover the case when the coefficients of the equation (1) are, generally speaking, non-smooth functions.

Note that the optimal control problem (1)-(3) is investigated using the following 3D integral representation in space \( W^{2,2,2}_p(G) \) [1]:

\[
u(x) = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \sum_{i_3=0}^{1} (x_1 - x^0_1)^{i_1} (x_2 - x^0_2)^{i_2} (x_3 - x^0_3)^{i_3} D^1_0 D^2_1 D^3_3 u(x^0_1, x^0_2, x^0_3) +
\]

\[
+ \sum_{i_2=0}^{1} \sum_{i_3=0}^{1} (x_2 - x^0_2)^{i_2} (x_3 - x^0_3)^{i_3} \int_{x^0_1}^{x_1} (x_1 - \tau_1) D^1_0 D^2_1 D^3_3 u(\tau_1, x^0_2, x^0_3) d\tau_1 + \]
35L25, 35R09, 31B10.

AMS Subject Classification: 3D optimal control problem, 3D Goursat problem, 3D Manjeron generalized equation, higher order hyperbolic equations, 3D integral representations, integral equations methods.

Keywords: 3D optimal control problem, 3D Goursat problem, 3D Manjeron generalized equation, higher order hyperbolic equations, 3D integral representations, integral equations methods.

REFERENCES

[3] Akhiev S.S., On a sheme for constructing fundamental solutions of local and non-local boundary value problems and their represen-

We note that some optimal control problems with distributed parameters have been studied in the author’s papers [4, 6].
CONTRIBUTIONS OF MAGNETIC TYPE INTERACTIONS TO THE VECTOR MESON-NUCLEON COUPLING IN THE BOTTOM-UP APPROACH

SHAHIN MAMEDOV\textsuperscript{1,2,3}, SHAHNAZ TAGHIYEVA\textsuperscript{2}

\textsuperscript{1}Institute for Physical Problems, Baku State University, Baku, Azerbaijan
\textsuperscript{2}Theoretical Physics Department, Physics Faculty, Baku State University, Baku, Azerbaijan
\textsuperscript{3}Institute of Physics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan
e-mail: sh.mamedov62@gmail.com, shahnaz.ilqarzadeh.92@mail.ru

1. Introduction

As is known \cite{1,3}, there are two kinds of magnetic type interactions in vector-fermion fields interactions in the bulk of AdS space in holographic QCD: 1) a magnetic gauge coupling term $L^{(1)}$, which is known from ordinary QFT:

$$L^{(1)} = \frac{i}{2} k_1 \left\{ \bar{\Psi}_1 \Gamma^{MN} F_{MN} \Psi_1 + \bar{\Psi}_2 \Gamma^{MN} F_{MN} \Psi_2 \right\},$$

where the $F_{MN} = \partial_M A_N - \partial_N A_M$ is stress tensor of the vector field $A_M$. 2) In \cite{1} it was introduced one more interaction term $L^{(2)}$ for a fermion field with the scalar and vector or axial vector fields, which changes fermion’s chirality in the bulk and consequently contributes to the chiral symmetry breaking on the boundary:

$$L^{(2)} = \frac{i}{2} k_2 \bar{\Psi}_1 X \Gamma^{MN} F_{MN} \Psi_2 - \frac{i}{2} k_2 \bar{\Psi}_2 X \Gamma^{MN} F_{MN} \Psi_1.$$  

(2)

The vector current of nucleons can be calculated from the generating function $Z$ which is defined as an exponent of classical bulk action $S$:

$$Z_{AdS} = e^{iS_{int}}.$$  

(3)

Here the interaction action $S_{int}$ is the 5-dimensional integral of the interaction lagrangian terms (1) and (2):

$$L_{int} = L^{(1)} + L^{(2)}.$$  

(4)

For the soft-wall model $S_{int}$ is defined as following

$$S_{int} = \int_0^\infty d^5 x e^{-k^2 z^2} L_{int},$$  

(5)

while in the hard-wall case we have no extra factor $e^{-k^2 z^2}$ and integration carries out in the slice $0 < z \leq z_m$ of AdS space-time:

$$S_{int} = \int_0^{z_m} d^5 x L_{int}.$$  

(6)

The metric of AdS space-time is chosen in Poincare coordinates:

$$ds^2 = \frac{1}{z^2} \left( \eta_{\mu\nu} dx^\mu dx^\nu - dz^2 \right).$$  

(7)
$Z_{AdS}$ is holographically identified with the generating function $Z_{QCD}$ of the boundary QCD:

$$Z_{AdS} = Z_{QCD}$$  \hspace{1cm} (8)

and the vacuum expectation value of the nucleon vector current in the boundary QCD theory will be found applying this equality as below:

$$< J^\mu >_{QCD} = -\frac{1}{\delta} \frac{\delta Z_{AdS}}{\delta A^\mu_0} |_{A^\mu_0=0}.$$  \hspace{1cm} (9)

Here $A^\mu_0 = A_\mu (q, z = 0) = A_\mu (q)$ is the UV boundary value of the vector field ($A (z = 0) = 1$).

The formula (5) will produce the vector current $J_\mu (p', p) = G_A \bar{u} (p') \gamma_\mu u (p)$ and the vector form factor of nucleons $G_A$ will be obtained as an integrals over the $z$ direction of the bulk space-time. The $A^\mu_0$ is the source for the $J_\mu$ current. Sum of these Lagrangian terms define contributions of the interactions (1) and (2) into the coupling constants $g_{\rho NN}^{(1)nm}$ and $f_{\rho}^{nm}$, which in the hard-wall model case have following expressions [1]:

$$g_{\rho NN}^{(1)nm} = -2 \int_0^{z_m} \frac{dz}{z^3} V_0' (z) \left[ k_1 \left( f_{1L}^{(n)*} f_{1L}^{(m)} - f_{2L}^{(n)*} f_{2L}^{(m)} \right) + k_2 v(z) \left( f_{1L}^{(n)*} f_{1L}^{(m)} + f_{2L}^{(n)*} f_{2L}^{(m)} \right) \right],$$

$$f_{\rho}^{nm} = 4m_N \int_0^{z_m} \frac{dz}{z^3} V_0 (z) \left[ k_1 \left( f_{1L}^{(n)*} f_{1R}^{(m)} - f_{2L}^{(n)*} f_{2R}^{(m)} \right) + k_2 v(z) \left( f_{1L}^{(n)*} f_{1R}^{(m)} + f_{2L}^{(n)*} f_{2R}^{(m)} \right) \right].$$  \hspace{1cm} (10)

In the case of soft-wall model the expressions for the $g_{\rho NN}^{(1)nm}$ and $f_{\rho}^{nm}$ constants will have integration over the fifth $z$ coordinate ranges $0 < z < \infty$ and these expressions contain extra $e^{-k^2 z^2}$ factors [3].

2. PROFILE FUNCTIONS FOR FIELDS IN THE HARD- AND SOFT-WALL MODELS

In hard wall model the profile function for the vector field is [6]:

$$V (z) = \frac{z J_1 (m_\rho z)}{\sqrt{\int_0^{z_m} dzz |J_1 (m_\rho z)|^2}}.$$  \hspace{1cm} (11)

Fermion fields profile functions are expressed in terms of Bessel functions of first kind [1, 5–7]:

$$F_{1L} = Cz^{5/2} J_2 (M_n z), \hspace{1cm} F_{1R} = Cz^{5/2} J_3 (M_n z);$$

$$F_{2L} = -Cz^{5/2} J_3 (M_n z), \hspace{1cm} F_{2R} = Cz^{5/2} J_2 (M_n z),$$  \hspace{1cm} (12)

where boundary fermions were taken on mass-shell $|p| = M_n$. The normalization constant was found in [9] and is equal to

$$C = \frac{\sqrt{2}}{z_m J_2 (M_n z_m)}. \hspace{1cm} (13)$$

Profile function for the scalar $X$ field is [4, 5]

$$X = \frac{1}{2} am_q z + \frac{1}{2a} \sigma z^3. \hspace{1cm} (14)$$

Here $m_q = 0.0083$ GeV , $\sigma = (0.213)^3$ GeV$^3$ and $a = \sqrt{N_c} / (2\pi)$ [4]. In the earlier works for the $g_{\rho NN}^{(1)nm}$ and $f_{\rho}^{nm}$ constants the factor $a$ in the profile function (14) was not taken into account.

In the soft-wall model the profile function for the $n$-th normalized Kaluza-Klein mode of spinor field is expressed in terms of Laguerre polynomials $L_n^{(a)}$ [2, 8]:

$$f_{1L}^{(n)} (z) = n_{1L} (kz)^{2a} L_n^{(a)} (kz), \hspace{1cm} f_{1R}^{(n)} (z) = n_{1R} (kz)^{2a-1} L_n^{(a-1)} (kz).$$  \hspace{1cm} (15)
Parameter $\alpha$ is related to the 5-dimensional mass $M$ via $\alpha = M + \frac{1}{2}$. It relates the mass of the $n$-th mode $m_n$ to the number $n$ in the following $m_n^2 = 4k^2 (n + \alpha)$. The constants $n_{L,R}$ are found from the normalization condition and are equal to following ones [8, 9]:

$$n_{1L} = \frac{1}{k^{\alpha-1}} \sqrt{\frac{2\Gamma(n+1)}{\Gamma(\alpha+n+1)}}, \quad n_{1R} = n_{1L}\sqrt{\alpha+n}.$$  \hspace{1cm} (16)

$M$ is equal to $M = \frac{3}{2}$ and hence $\alpha = 2$.

For the $\rho$-meson we have the profile function for the vector field $V_n(z)$ becomes [2, 8, 9]

$$V_n(z) = k^2 z^2 \sqrt{\frac{2}{n+1}} L_n^1 (k^2 z^2).$$  \hspace{1cm} (17)

In the soft-wall model the scalar field $X$ has same profile function as in the hard-wall case.

The numerical calculations for the constants $f$ and $g$ are presented in the Table 1 below.

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$g_{\rho NN}^{(1) \text{SW}}$</th>
<th>$g_{\rho NN}^{(1) \text{HW}}$</th>
<th>$f_{\rho NN} \text{SW}$</th>
<th>$f_{\rho NN} \text{HW}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.78</td>
<td>0.5</td>
<td>1.39</td>
<td>0.053</td>
<td>4.25</td>
<td>0.18</td>
</tr>
<tr>
<td>-0.98</td>
<td>1.25</td>
<td>3.11</td>
<td>0.078</td>
<td>10.25</td>
<td>0.25</td>
</tr>
</tbody>
</table>

**Keywords:** AdS/QCD, meson, nucleon, coupling constant.

**References**

FINDING THE GUARANTEED SUBOPTIMAL SOLUTION TO THE FUNCTIONAL IN THE INTEGER PROGRAMMING PROBLEM

K.SH. MAMMADOV¹, N.N. MAMMADOV¹

¹Baku State University, Baku, Azerbaijan
e-mail: mamedov_knyaz@yahoo.com, nazim.484.99@gmail.com

Abstract. In this work the integer programming problem is considered. The method for finding the guaranteed suboptimal solution is suggested.

Keywords: Guaranteed suboptimal solution, integer programming problem, method, mathematical model.

AMS Subject Classification: 35Q90.

Let’s consider the integer programming problem as below:

\[ \sum_{j=1}^{n} c_j x_j \rightarrow \max, \]

(1)

\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad (i = 1, m), \]

(2)

\[ 0 \leq x_j \leq d_j, \quad (j = 1, n). \]

(3)

Here \( c_j > 0, \quad a_{ij} \geq 0, \quad d_j > 0, \quad b_i > 0, \quad (i = 1, m; \quad j = 1, n) \) are the given integer numbers.

We have introduced the guaranteed solution and the guaranteed suboptimal solution for the problem (1)-(3). The method for finding the guaranteed suboptimal solution is suggested.

As it is known, this problem belongs to the class of difficult solvable problems, i.e., to the NP-complete class. In other words, there are no methods of polynomial time complexity for finding the optimal solution. On the other hand, for an optimal solution of this problem, there are the methods “intrusions and boundaries”, “dynamic programming” and “combinator” [1–3,9,10,12]. However, these methods can not solve the problem in real time at large values of a number \( n \). Therefore algorithms for rapid calculation (i.e., polynomial time complexity) have been developed for construction of suboptimal (approximate) solutions which significantly don’t differ from the optimal solution [3, 4, 9–11]. Suppose that using one of the known methods an suboptimal solution \( X^s = (x_1^s, x_2^s, \ldots, x_n^s) \) and corresponding value of the function (1)

\[ f^s = \sum_{j=1}^{n} c_j x_j^s \]
have been found. Then we can define the certain number $\Delta^s = \left[ f^s \frac{p}{100} \right]$. Here the parameter $p$ is the increasing percentage of the value $f^s$, the sense $[\cdot]$ is shows full part of the $z$. Let’s we want to get more income without changing the giving known costs $b_i$, $(i = 1, m)$ and the resources $a_{ij}$ $(i = 1, m, j = \overline{1, n})$. Obviously, to achieve this goal the market prices $c_j$ $(j = \overline{1, n})$ should be increased or decreased at indicated intervals. The changes in the prices of sale must be made so that new income was not less than the $f^s + \Delta^s$. If initial maximum income should be increased at least $p\%$, then it can be chosen as $\Delta^s = \left[ f^s \frac{p}{100} \right]$. Therefore, we have to add such minimal numbers on the coefficients of the function (1) in the integer interval $[\alpha_j, \beta_j]$, $(j = \overline{1, n})$ without changing the $a_{ij}$ $(i = 1, m, j = \overline{1, n})$ costs and the allocated resources $b_i$, $(i = 1, m)$, so that the obtained solution is not less than the parameter $f^s + \Delta^s$ of the values of function (1). Here, the conditions $\alpha_j \leq 0$ and $\beta_j \geq 0$, $(j = \overline{1, n})$ must be satisfied. Note that in [5-8] some cases of this problem were considered and the corresponding solution methods were developed.

It should be noted that the solution of such problems in the context of financial difficulties is relevant. So, we get a new mathematical model as follows:

\[
\sum_{j=1}^{n} (c_j + \delta_j)x_j \rightarrow \max, \quad (4)
\]

\[
\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad (i = \overline{1, m}), \quad (5)
\]

\[
0 \leq x_j \leq d_j, \quad (j = \overline{1, n}), \quad (6)
\]

\[
\alpha_j \leq \delta_j \leq \beta_j, \quad (j = \overline{1, n}). \quad (7)
\]

Here $\delta_j$ $(j = \overline{1, n})$ is integer number and the conditions $\alpha_j \leq 0$, $\beta_j \geq 0$, $(j = \overline{1, n})$ must be satisfied. Since if $\delta_j \in [\alpha_j, 0]$ $(j = \overline{1, n})$, the corresponding values $c_j$ $(j = \overline{1, n})$ should be decreased by $\delta_j$ $(j = \overline{1, n})$, if $\delta_j \in [0, \beta_j]$ $(j = \overline{1, n})$, then the corresponding values $c_j$ $(j = \overline{1, n})$ should be increased $\delta_j$ $(j = \overline{1, n})$. On the other hand, the function (4) is nonlinear, since there exist the products $\delta_jx_j$ $(j = \overline{1, n})$. This makes difficult to solve the problem (4)-(7). So, we get the following mathematical model based on the problem statement:

\[
\delta_j \rightarrow \min, \quad (j = \overline{1, n}) \quad (8)
\]

\[
\sum_{j=1}^{n} (c_j + \delta_j)x_j \geq f^s + \Delta^s, \quad (9)
\]

\[
\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad (i = \overline{1, m}), \quad (10)
\]

\[
\alpha_j \leq \delta_j \leq \beta_j, \quad (j = \overline{1, n}), \quad (11)
\]

\[
0 \leq x_j \leq d_j, \quad (j = \overline{1, n}). \quad (12)
\]

Here $c_j > 0$, $a_{ij} \geq 0$, $d_j > 0$, $\alpha_j \leq 0$, $\beta_j \geq 0$, and $b_i > 0$ $(i = \overline{1, m}, j = \overline{1, n})$ are giving constant integer numbers.

The guaranteed suboptimal solution of the problem (8) - (12) to the functional is called $n$ - dimensional solution $X^2 = (x_1^2, x_2^2, \ldots, x_n^2)$, that gives small value to the parameter $\delta_j$ $(j = \overline{1, n})$.
For solving this problem the method is proposed. The convergence of the method is based on the following theorem:

**Theorem 1.** If for any \( j (j = \overline{1,n}) \) the relations \( |\beta_j^k - \alpha_j^k| \leq 1 \) in a certain \( k^{th} \) stage are satisfied then the solutions \( X_{s;k} = (x_{s;1}^k, x_{s;2}^k, \ldots, x_{s;n}^k) \) is the guaranteed suboptimal solution of the problem (1) - (3).

Here the numbers \( \alpha_j^k \) and \( \beta_j^k \) \( (j = \overline{1,n}) \) are the changed values correspondingly giving \( \alpha_j \) and \( \beta_j \), \( (j = \overline{1,n}) \) numbers at the \( k \)th stage.

**References**


METHODS FOR EVALUATION OF HUMAN RESOURCES
PERFORMANCE IN VIRTUAL ORGANIZATIONS

M.H. MAMMADOVA¹, Z.G. JABRAYILOVA¹

¹Institute of Information Technology of Azerbaijan National Academy of Science, Baku
e-mail: depart15@iit.ab.az

1. INTRODUCTION

The massive spread of Internet technologies and the globalization of the economy in the late twentieth century have allowed the enterprises to form temporary partnerships, and subsequently have led to the formation and expansion of virtual organizations (VO) by establishing such links over the Internet [6]. Consequently, virtual firms, virtual corporations, virtual departments, virtual groups, and virtual working spectrum have enlarged in the sphere of employment. The process of virtualization of organizations has changed the social-labor relations and traditional governance mechanisms in the labor sphere, which has led to the change of staffing functions, the content of labor contracts and the change of traditional institutions for information collection. Unquestionably, the success of VO depends on appropriate organization of human resources management (HRM) and selection, deployment, socialization, motivation and evaluation of employees and partners. In this regard, this paper analyzes the features of HRM in VO and proposes a method for the assessment of the performance of virtual employees (contractors).

2. FEATURES OF HRM IN VO

In each organization, the goal of HRM is to form, employ and develop human resources [2, 3, 4].

The use of human resources in VO envisages the evaluation, rewarding, and motivation of each employee in achieving efficient production and common outcome.

The characteristics of VO may include dynamics, informal communication, multidisciplinary teams, uncertainty of organizational boundaries, objective-oriented, working from home, lack of visible organizational structure, reference to mental work of integrated employees, etc., which define the following specific features of HRM issues:

1. incomplete and inaccurate information provided to the decision-maker about the capabilities of companies, co-workers and partners included into VO and about their performance;
2. establishment of relationships of VO’s operation on the basis of trust factor, and consequently, dependence on the results of the operation on individual and psychological aspects;
3. dependence of general corporate outcome on local results, and the latter’s inaccuracies and uncertainty;
4. dependence of the results of VO HRM issues on quantitative and qualitative aspects, and the emergence of uncertainties arising from the time factor;
(5) uncertainty emerged in the assessment of the performance of a virtual employee or partner, etc.

These aspects characterize HRM issues in VO, particularly selection and evaluation of employees as poorly structured and hard-to-formalize issues, and necessitate the use of fuzzy logic theory [5, 7].

3. USE AND ASSESSMENT OF HUMAN RESOURCES IN VO

Performance of VO’s staff is "short-term", and therefore, the lack of mechanisms to measure each employee’s contribution to the achievement of the overall objective and to stimulate "short-term” activities is one of the key issues to be solved in VO [1, 2].

The use of human resources in VO is accompanied by a variety of goals. Since VO is a temporary organization consisting of different employees (groups, companies) performing independent functions and different objectives for problem solution, unsurprisingly their interests in achieving a common goal will be different, and assessing the outcome of each employee’s performance is topical. Assessment of the performance of virtual groups (or corporations) in VO depends on the nature of their work. Thus, assessment of the work of virtual groups performing the same function can be essentially characterized as the assessment of the objects that are categorized by the same criteria and functioning in a fragmented environment. If virtual groups perform different functions, their activities can be characterized as the assessment of the objects that are categorized by the same criteria and functioning in a fragmented environment. Abovementioned determine the dependence of the overall corporate outcome on the local results and the latter’s inaccuracy and uncertainty. From this point of view, the evaluation of employees in VO is characterized as a system of fragmented subsystems, i.e., VO consists of several groups of identically organized network nodes [6, 7]. Each of these groups has individual objective, however does not have sufficient information and resources to solve the common problem. Depending on the nature of the function performed by the virtual groups, three types of distribution are distinguished: 1) virtual groups are performing the same functions - horizontal distribution; 2) virtual groups are performing the functions of different characteristics - vertical distribution; 3) virtual groups are performing the functions of blended characteristic - blended distribution. The physical fragmentation of the needed information depends on the criteria that characterize the alternatives (employees) and the relations with the sub-criteria. Using the fuzzy logic apparatus, the methods are offered for decision-making in a distributed environment for the assessment of employees' performance in VO and a fuzzy relay model is used for this purpose.

Decision-making methods in distributed environments. The following conditions in horizontal distribution of Virtual Groups are ensured: $X = \bigcup_{g=1}^{G} X_{ig}$, where $X$ - is a set of alternatives, $X_{ig}$-alternatives in the $g$-th fragment (sub-system) and $X_{ig}\cap X_{ij} = \emptyset$ and $K_g\cap K_j = K_j = K$ for $\forall g\neq j$. That is, the alternatives distributed by fragments are characterized by the same criteria [3, 4]. Criteria have a hierarchical structure and different weight coefficients, i.e.:

$$K = \{K_m, \ m = 1, M\}, \ w_1, w_2, \ldots, w_T \ - \ are \ the \ relative \ importance \ ratios \ of \ criteria \ K_m, m = 1, M.$$  

$$K_m = \{k_m, t = 1, T\} \ w_{m1}, w_{m2}, \ldots, w_{mT} \ - \ are \ the \ relative \ importance \ ratios \ of \ sub-criteria \ characterizing \ the \ criterion \ K_m.$$  

Assume that, in each fragment, the membership function of sub-criteria is known:

$$\{\varphi_{k_{m1}} (x_{ig}), \varphi_{k_{m2}} (x_{ig}), \ldots, \varphi_{k_{mT}} (x_{ig})\} = \{\varphi_{k_{mt}} (x_{ig}), \ t = 1, T\}.$$  

Decision making process is performed on the following steps:

1. The membership function of alternatives to the criterion $K_m$ in each fragment is defined:

$$\varphi_{K_m} (x_{ig}) = \sum_{t=1}^{T} w_{mt} \varphi_{k_{mt}} (x_{ig}).$$
2. The membership function of alternatives to the generalized criterion $K$ in each fragment is defined: \[ \varphi_K(x_{ig}) = \sum_{m=1}^{M} w_m \varphi_{K_m}(x_{ig}) \].

3. The maximum of alternatives by fragments is selected: \( \varphi(x^*) = \max \varphi_K(x_{ig}) \), \( g = 1, G \), \( i = 1, N \). The alternative with the maximum value is the final decision on horizontally distributed virtual groups, and it is found out of set of decisions with the maximum values on fragments, i.e., \( x^* \in \{ x_{ig}, g = 1, G \} \). The following conditions are ensured for vertical distribution of virtual groups: \( K = \bigcup_{m=1}^{M} K_m \), where \( K \) consists of a set of criteria \( K = \{ K_m, m = 1, M \} \), with different importance degree, and \( M \) — the number of fragments (groups). \( K_m = \{ k_{mt}, t = 1, T \} \) is a set of sub-criteria with different importance degree in each fragment, and \( w_{m1}, w_{m2}, \ldots, w_{mT} \) are the relative importance factors of sub-criteria.

In this case, the following conditions are ensured: \( K_{mg} \cap K_{mj} = \emptyset \) and \( X_g \cap X_j = X_g = X_j = X \) for \( \forall g \neq j \). That is, the same alternatives for vertically distributed VO by fragments are characterized and evaluated by different criteria [3, 4]. Assume that membership function of the alternative \( x_i \) to the sub-criteria in the \( m \)-th fragment is known: \( \{ \varphi_{km1}(x_i), \varphi_{km2}(x_i), \ldots, \varphi_{kmT}(x_i) \} = \{ \varphi_{km}(x_i), t = 1, T \} \).

The final decision on the employees’ performance in the vertically fragmented VO is made as follows:

1. The membership function of alternatives to the criterion \( K_m \) in each fragment is defined:
   \[ \varphi_{Km}(x_i) = \sum_{t=1}^{T} w_{mt} \varphi_{mt}(x_i) \].

2. The membership function of alternatives to the generalized criterion is defined:
   \[ \varphi_K(x_i) = \sum_{m=1}^{M} w_m \varphi_{Km}(x_i) \].

3. The alternative with the maximum value for the membership function to the generalized criterion \( K \) is chosen: \( \varphi_K(x^*) = \max \{ \varphi_K(x_i), i = 1, N \} \).

Selected alternative with the maximum value is corresponding to the final decision on the vertical fragments, and this decision may not be from the best decisions, i.e., from the set of fragments \( \varphi_{Km}(x^*) = \max \{ \varphi_{Km}(x_i), i = 1, N \} \).

In an environment where the VO is distributed by blended (both horizontally and vertically) fragments, the decision-making process can be accomplished by referencing to decision-making methods in a vertical distribution environment.

**Keywords:** Virtual organization, human resources management, fuzzy environment, virtual employees, decision-making methods in distributed environment.

**AMS Subject Classification:** 68U35, 90B50.

**References**


MULTIPOINT NECESSARY OPTIMALITY CONDITIONS OF SINGULAR CONTROLS IN DELAYED STOCHASTIC SYSTEMS

K.B. MANSIMOV\textsuperscript{1,2}, R.O. MASTALIYEV\textsuperscript{2}

\textsuperscript{1} Baku State University, Baku, Azerbaijan
\textsuperscript{2} Institute of Control Systems of ANAS, Baku, Azerbaijan
e-mail: kamilbmansimov@gmail.com, mastaliyvrashad@gmail.com

In the report we consider a problem of optimal control of the Ito stochastic systems with delayed argument \cite{5,6}. Stochastic analog of Pontryagin’s maximum principle is derived, singular controls \cite{4} is studied and new necessary conditions of optimality for singular controls are obtained.

Let \((\Omega, \mathcal{F}, P)\) - be a complete probability space with definition in it non-decreasing flow of \(\mathcal{F}_t\), where \(\mathcal{F}_t = \sigma(w(s), t_0 \leq s \leq t)\), and \(w(t)\) is \(n\)-dimensional standard Wiener process.

\[ L^2_{\mathcal{F}}[t_0, t_1; \mathbb{R}^r] \] - is the space of measurable with respect to \((t, \omega)\) random processes \(x(t, \omega)\) : \(T = [t_0, t_1] : \Omega \rightarrow \mathbb{R}^n\), for which \(E \int_{t_0}^{t_1} \|x(t)\|^2 dt < \infty\). \(E\) is the sign of mathematical expectation.

Assume that it is required to find the minimum of the functional

\[ I(u) = E\{\varphi(x(t_1))\}, \]  

under the constraints

\[ u(t, \omega) \in U_d \equiv \{u(., .) \in L^2_{\mathcal{F}}(t_0, t_1; \mathbb{R}^r)/u(., .) \in U \subset \mathbb{R}^r\}, \]  

\[ dx(t) = f(t, x(t), x(h(t)), u(t))dt + \sigma(t, x(t), x(h(t)))dw(t), \quad t \in (t_0, t_1], \]  

\[ x(t) = \Phi(t), \quad t \in E_{t_0} = [h(t_0), t_0), \]  

\[ x(t_0) = x_0. \]  

Here \(\varphi(x)\) - is the twice continuously differentiable scalar function, \(f(t, x, y, u)\) - is a given \(n\)-dimensional vector-function continuous in totality of variables together with partial derivatives with respect to \((x, y)\) to the second order inclusively, where \(y(t) = x(h(t)), h(t) < t\) - is the given continuously differentiable scalar function, where \(h(t) > 0, \sigma(t, x, y) : T \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}\) - is a matrix function continuous in totality of variables together with partial derivatives with respect to \((x, y)\) to the second order inclusively, \(\Phi(t) \in L^2_{\mathcal{F}}(h(t_0), t_0; \mathbb{R}^n)\) - is almost surely continuous in \(E_{t_0}\) initial vector-function.

It is assumed that to each admissible control \(u(t)\) there corresponds a unique solution \(x(t)\) of the system (3)-(4) with almost sure continuous trajectories determined on \(T\).

Note that earlier in particular cases (for \(h(t) = t - \tau\), and \(h(t) = t - \gamma(t)\)) for such optimal control problems, necessary optimality conditions are obtained in the form of Pontryagin’s maximum principle in \cite{1-3}, singular controls were considered in \cite{1-2}. 

\cite{261}
In this paper, by means of the stochastic analog of the method suggested in K.B. Mansimov’s works \cite{7,8}, etc., the multipoint necessary conditions of optimality of singular controls that corner also the earlier obtained results, are obtained.

We introduce the following notation:

\[ H(t, x(t), y(t), u(t), \psi(t)) = \psi'(t)f(t, x(t), y(t), u(t)), \]
\[ f_x[t] = f_x(t, x(t), y(t), u(t)), \]
\[ H_x[t] = H_x(t, x(t), y(t), u(t), \psi(t)), \]
\[ H_{xx}[t] = H_{xx}(t, x(t), y(t), u(t), \psi(t)), \]
\[ \sigma_x[t] = \sigma_x(t, x(t), y(t)), \]
\[ \Delta_vf[t] = f(t, x(t), y(t), v) - f(t, x(t), y(t), u(t)), \]
\[ \Delta_vH[t] = H(t, x(t), y(t), v, \psi(t)) - H(t, x(t), y(t), u(t), \psi(t)). \]

Here \( \psi(t) \in L^2_{F}(t_0, t_1; R^n) \), \( \beta(t) \in L^2_{F}(t_0, t_1; R^{n \times n}) \) are solutions of the system of stochastic differential equations (adjoint system):

\[
\begin{align*}
\frac{d\psi(t)}{dt} &= -(H_x[t] + (H_y[r(t)] + \beta(r(t))\sigma_y[r(t)])\dot{r}(t))dt + \beta(t)dw(t), \quad t \in [t_0, h(t_1)), \\
\frac{d\psi(t)}{dt} &= -(H_x[t] + \beta(t)\sigma_x[t])dt + \beta(t)dw(t), \quad t \in [h(t_1), t_1), \\
\psi(t_1) &= -\varphi_x(x(t_1)),
\end{align*}
\]

where \( r(t) \) is the inverse of \( h(t) \).

We known (see, for examp.\cite{1-3}) that for the optimality of the admissible control \( u(t) \) in the problem \((1) - (4)\) it is necessary that the inequality

\[ E\Delta_vH[\theta] \leq 0, \]

fulfills for all \( v \in U, \theta \in [t_0, t_1) \), here \( \theta \) is an arbitrary Lebesgue point of control \( u(t) \).

This inequality is a Pontryagin maximum condition for the under consideration problem.

It is clear that the maximum condition \((5)\), being first order necessary optimality condition, gives limited information about the control suspicious of optimality. In addition, is not ruled out possibility of degeneracy of the Pontryagin maximum condition. Let us consider degeneracy cases of the Pontryagin maximum principle.

**Definition** \cite{4}. The admissible control \( u(t) \), we call a singular control, in the sense of the Pontryagin maximum principle, if for all \( v \in U, \theta \in [t_0, t_1) \),

\[ \Delta_vH[\theta] = 0. \]

Suppose that the fundamental matrix \( F(t, \tau) \) is a solution of the homogeneous equation:

\[
\begin{align*}
\frac{dF(t, \tau)}{dt} &= (f_x[t]F(t, \tau) + f_y[t]F(h(t), \tau))dt + (\sigma_x[t]F(t, \tau) + \sigma_y[t]F(h(t), \tau))dw(t), \\
F(t, t) &= I, \quad I – unit matrix, \\
F(t, \tau) &= 0, \quad \tau > t.
\end{align*}
\]

We set

\[ K(\tau, s) = -F(t_1, \tau)\varphi_{xx}(x(t_1))F(t_1, s) + \]
\[ + \int_{\max(\tau, s)}^{t_1} [F(t, \tau) H_{xx}[t]F(t, s) + F(t, \tau) H_{xy}[t]F(h(t), s) + F(h(t), \tau) H_{yx}[t]F(t, s) + \\
F(h(t), \tau) H_{yy}[t]F(h(t), s)] dt. \]
Theorem. For the optimality of a singular control, in the sense of the Pontryagin maximum principle \( u(t) \) in problem (1) - (4) it is necessary that for any natural number \( m \) the inequality
\[
E\left[ \sum_{i=1}^{m} \sum_{j=1}^{m} l_i l_j \Delta_{v_i} f[\theta_i] K(\theta_i, \theta_j) \Delta_{v_j} f[\theta_j] + \sum_{i=1}^{m} l_i \Delta_{v_i} H_x[\theta_i] [l_i \Delta_{v_i} f[\theta_i]] + 2 \sum_{j=1}^{m} l_j F(\theta_i, \theta_j) \Delta_{v_j} f[\theta_j] \right] + 
2 \sum_{i=1}^{m} l_i \Delta_{v_i} H_y[\theta_i] \sum_{j=1}^{m} l_j F(\theta_i, \theta_j) \Delta_{v_j} f[\theta_j] \right] \leq 0,
\]
satisfied for all \( l_i \geq 0, v_i \in U, \theta_i \in [t_0, t_1), i = \overline{1, m}, t_0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_m < t_1 \).

Inequality (6) is a sequence necessary optimality conditions for singular controls, in the Pontryagin maximum principle sense, and allows us to significantly narrow the set of singular controls suspicious of optimality.

From the inequality (6) in particular, follows simpler and more constructive the necessary optimality conditions. But they are less informative than (6).

Here are some of them.

For \( m = 1 \) we get

**Corollary 1.** If \( u(t) \) is a singular optimal control in problem (1) - (4), then along the process \((u(t), x(t))\) for all \( v \) and \( \theta \in [t_0, t_1), \)
\[
R(\theta, v) = E\{ \Delta_v f[\theta] K(\theta, \theta) \Delta_v f[\theta] + \Delta_v H_x[\theta] \Delta_v f[\theta] \} \leq 0.
\]

For \( m = 2 \) we get

**Corollary 2.** If \( u(t) \) is a singular, optimal control in the problem (1) - (4), then the following relations are satisfied along the process \((u(t), x(t))\):
\[
ER(\theta_1, v_1) \leq 0, ER(\theta_2, v_2) \leq 0,
\]
\[
E\{ \Delta_{v_2} H_y[\theta_2] F(\theta_2, \theta_1) \Delta_{v_1} f[\theta_1] + \Delta_{v_1} f[\theta_2] K(\theta_2, \theta_1) \Delta_{v_1} f[\theta_1] \} \leq \sqrt{ER(\theta_1, v_1)ER(\theta_2, v_2)},
\]
for all \( v_1, v_2 \in U, \theta_1, \theta_2 \in [t_0, t_1), (\theta_1 \leq \theta_2) \).

**Keywords:** Stochastic control problem, necessary optimality conditions, singular controls, multipoint necessary optimality conditions.

**AMS Subject Classification:** 93E20.

**References**

TO SUFFICIENT CONDITIONS FOR STRONG EXTREMUM IN CALCULUS OF VARIATIONS

MARDANOV MISIR¹, MELIKOV TELMAN², MALIK SAMIN³

¹Institute of Mathematics and Mechanics of ANAS, Baku, Azerbaijan
²Institute of Mathematics and Mechanics of ANAS, Institute of Control Systems, Baku, Azerbaijan
³Institute of Mathematics and Mechanics of ANAS, Baku Higher Oil School, Baku, Azerbaijan
e-mail: misirmardanov@yahoo.com, t.melik@rambler.ru, saminmelik@gmail.com

1. INTRODUCTION AND SETTING OF THE PROBLEM

Note that the Weierstrass’s known method suggested by him in 1879. It allows to study the classical variational problem by means of fields of extremals. However, this method did not get more widespread as a constructive means of analysis of variation problems in connection with difficulties of construction of the field along the domain under considerations. Therefore, obtaining of a more constructive sufficient condition for strong minimum is important both from theoretical and practical point of view.

Let us consider the simplest problem in calculus of variation

\[ J(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt \rightarrow \inf_{x(\cdot)} \]

\[ x(t_0) = x_0, \quad x(t_1) = x_1, \] (1)

where \( x(t) \in \mathbb{R}^n \), \( t \in [t_0, t_1] \subset \mathbb{R} \), \( x_0, x_1, t_0, t_1 \) are given points; \( L(t, x, \dot{x}) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) is the given function.

The problem (1)-(2) is studied in the case when \( x(\cdot) \) belongs to the space \( KC^1([t_0, t_1], \mathbb{R}^n) \) of functions with a piecewise-continuous derivative provided with the norm \( \|x(\cdot)\|_{C([t_0, t_1], \mathbb{R}^n)} \).

The functions \( x(\cdot) \in KC^1([t_0, t_1], \mathbb{R}^n) \), satisfying the boundary conditions (2) are called admissible.

Here we assume the following:

\[ L(t, x, \dot{x}) = L^{(0)}(t, \dot{x}) + 2x^T A(t) \dot{x} + x^T B(t)x + g^T(x) \dot{x} + p^T(t)x, \] (3)

where \( L^{(0)}(t, \dot{x}) \in C([t_0, t_1] \times \mathbb{R}^n, \mathbb{R}) \), \( g(x) \in C(\mathbb{R}^n, \mathbb{R}^n) \), \( p(t) \in C([t_0, t_1], \mathbb{R}^n) \); \( A(\cdot), B(\cdot) \) are \( n \times n \) matrices, \( A^T(t) = A(t) \), \( A(t) \in C^1([t_0, t_1]) \), \( B(t) \in C([t_0, t_1]) \).

In the paper, for a particular case of problem (1)-(2), we formulate a sufficient condition for a strong extremum that allows us to overcome the above-mentioned difficulties.
2. SUFFICIENT CONDITION FOR STRONG MINIMUM

In this paper, we obtained a sufficient condition for the absolute minimum in the simplest problem of the calculus of variation (1)-(2).

Considering assumption (3) let us show that along an admissible function \( \pi(\cdot) \) for increment \( \Delta J(\tilde{\pi}(\cdot)) = J(\pi(\cdot) + \Delta x(\cdot)) - J(\tilde{\pi}(\cdot)) \) of functional (1) the next formula is valid for arbitrary admissible function \( \tilde{x}(\cdot) := \pi(\cdot) + \Delta x(\cdot) \):

\[
\Delta J(\tilde{\pi}(\cdot)) = \int_{t_0}^{t_1} \left\{ L^{(0)}(t, \dot{\tilde{x}}(t) + \Delta \dot{x}(t)) - L^{(0)}(t, \dot{\tilde{x}}(t)) + \right.
\]
\[
+ \int_{t_0}^{t_1} \left[ p^T(\tau) + \tilde{x}(\tau)(B(\tau) + B^T(\tau) - 2\dot{A}(\tau)) \right] d\tau \Delta \dot{x}(t) +
\]
\[
+ \Delta x(T)(B(T) - \dot{A}(T))\Delta x(T) \right\} dt.
\]  

(4)

Here, the following identities are considered:

\[
\int_{t_0}^{t_1} [g(\tilde{x}(t))\dot{x}(t) - g(\pi(t))\dot{\pi}(t)] dt = 0,
\]
\[
2 \int_{t_0}^{t_1} [\tilde{x}(t)A(t)\dot{x}(t) - \pi^T(t)A(t)\dot{\pi}(t)] dt = - \int_{t_0}^{t_1} [\tilde{x}(t)\dot{A}(t)\dot{x}(t) - \pi^T(t)\dot{A}(t)\pi(t)] dt,
\]
\[
\int_{t_0}^{t_1} p^T(\tau)\Delta x(T) dt = \int_{t_0}^{t_1} (\int_{t_0}^{t_1} p^T(\tau) \Delta x(T) dt) dt.
\]

By (4) and some reasonings, the next theorem is proved.

**Theorem 1.** Let assumption (3) be fulfilled and the matrix \( B(t) - \dot{A}(t) \) be nonnegative-definite. Moreover, suppose that there exists an admissible function \( \tilde{x}(\cdot) \) and some vector \( \lambda \in \mathbb{R}^n \) such that the inequality

\[
L^{(0)}(t, \dot{\tilde{x}}(t) + \xi) +
\]
\[
+ \{ \lambda^T + \int_{t_0}^{t_1} [p^T(\tau) + \tilde{x}(\tau)(B(\tau) + B^T(\tau) - 2\dot{A}(\tau)) \} \xi \geq L^{(0)}(t, \dot{\tilde{x}}(t)),
\]
\[
\forall \xi \in \mathbb{R}^n, \ a.e. \ t \in [t_0, t_1]
\]  

(5)

be fulfilled.

Then the admissible function \( \tilde{x}(t), t \in [t_0, t_1] \) affords an absolute minimum in the problem (1)-(2).

It should be noted that in contrast to the earlier known sufficient conditions for strong extremum (see e.g. [1], [2]), the proof of Theorem 1 is obtained without assumptions on smoothness of the integrant \( L(t, x, \dot{x}) \) with respect to the variables \( x, \dot{x} \).

In addition, we note that, for instance, the following example shows the effectiveness of Theorem 1 more clearly

**Example.**

\[
J(x(\cdot)) = \int_{-1}^{1} (t^2 \dot{x}^2 + 12x^2) dt \to \min_{x(\cdot)},
\]
\[
x(-1) = -1, \ y(1) = 1.
\]  

(6)

(7)

Let us investigate the problem (6)-(7). The Euler-Lagrange equation gets the form:

\[
t^2 \ddot{x} + 2t \dot{x} - 12x = 0
\]  

(8)

The general solution of (8) is \( x(t) = c_1 t^3 + c_2 t^{-4} \).

It is clear that \( \dot{x}(t) = t^3 \) is an admissible extremal, and it cannot be covered with a field of extremals. Consequently, in the study of the problem (6)-(7), sufficient condition of Weierstrass...
strong minimum cannot be applied. Also note that the Weierstrass’s necessary condition is satisfied:

\[ E(t, \hat{x}(t), \dot{\hat{x}}(t)) = t^2 \xi^2 \geq 0, \quad t \in [-1, 1], \quad \xi \in R. \]

The strengthened Legendre condition is not satisfied:

\[ L_{x\dot{x}}(t, \hat{x}(t), \dot{\hat{x}}(t)) = 2t^2 > 0, \quad t \in [-1, 1]. \]

Thus these conditions leave the admissible extremal \( \hat{x}(t) = t^3 \) is still a contender for extremum. Now let us apply the Theorem 1 to solve (6)-(7). Here \( L^{(0)}(t, \hat{x}) = t^2 \dot{x}^2, \quad A(t) = p(t) = 0, \quad B(t) = 12. \) Then, let us show that there exist \( \lambda \in R \) and admissible function \( \hat{x}(t) \) such that the following inequality holds:

\[ t^2(\hat{x}(t) + \xi)^2 + (\lambda + 24 \int_t^1 \tau(x) d\tau) \xi \geq t^2 \hat{x}^2, \quad \xi \in R, \quad a.e. \ t \in [-1, 1]. \] (9)

In order for inequality (9) to be valid, it is necessary and sufficient that the following holds:

\[ 2t^2 \hat{x}(t) + 24 \int_t^1 \tau d\tau + \lambda = 0, \quad a.e. \ t \in [-1, 1], \] (10)

where \( \hat{x}(-1) = \hat{x}(1) = 1. \)

Here, we get that the admissible function \( \hat{x}(t) \) is the solution of Euler-Lagrange equation, i.e.

\[ t^2 \hat{x}(t) + 2t \dot{\hat{x}}(t) + 12 \hat{x}(t) = 0, \]

\[ \hat{x}(-1) = -1, \quad \hat{x}(1) = 1. \] (11)

Solving (11) we get \( \hat{x}(t) = \dot{x}(t) = t^3, \quad t \in [-1, 1]. \) Taking this into account in (10) we obtain \( \lambda = -6. \) Therefore, by Theorem 1, we state that \( \hat{x}(t) = t^3 \) is an absolute minimum in the problem (6)-(7).

In conclusion, we note that an analogue of Theorem 1 can be obtained for more complicated problem of the calculus of variations.

**Keywords:** Extremal, extremum of functional, strong minimum, sufficient condition.

**AMS Subject Classification:** 58E30.

**References**


TRAPEZOIDAL QUADRATURE RULE FOR SOLVING NONLINEAR FUZZY FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS OF THE HAMMERSTEIN TYPE

A. MASHHADI Gholam¹, R. EZZATI¹, T. ALLAHVIRANLOO²

¹Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran
²Department of Mathematics, Science and Research Branch
Islamic Azad University, Tehran, Iran
e-mail: ezati@kiau.ac.ir

ABSTRACT. The aim of this paper is to solve nonlinear fuzzy fractional integro-differential equations (NFFIDEs) of the Hammerstein type. The basic idea is to convert fuzzy fractional integro-differential equations to a type of second kind fuzzy Volterra integral equations. Then by using some suitable quadrature rules, the fuzzy Volterra integral equations is obtained. The convergence of the method is proved.

Keywords: Nonlinear fuzzy fractional integro-differential equations, fuzzy Volterra integral equations.

AMS Subject Classification: 34A12.

1. Introduction

The subject of fractional calculus has a long history and spread like wildfire during the past three decades and has been found wide applications in many areas of science and engineering. Fuzzy set theory is a powerful tool for modeling uncertain problems. Therefore, large varieties of natural phenomena have been modeled using fuzzy concepts. The operations of fuzzy fractional calculus are more complicated then the classical one, and their application in solving fuzzy fractional equations is also more difficult then in the integer order case. The present work is obtained approximate solutions for nonlinear fuzzy fractional Volterra integro-differential equations of the Hammerstein type.

\[
\begin{align*}
\mathfrak{c}_H (D^\alpha y)(x) &= g(x) + \int_0^x k(x, t) \odot G(y(t)) dt, \ x \in [0, b], \ n - 1 < \alpha \leq n, \\
\text{subject to the initial conditions} \\
y^{(i)}(0) &= \delta_i, \ i = 0, 1, \cdots, n - 1,
\end{align*}
\]

(1)

where \(\delta_i\) are given constants.

Numerical methods for approximating fuzzy integral, fuzzy fractional integral and derivatives have been studied by many authors. For example [2–5].
The structure of this paper is as follows: In Section 2, we introduce preliminaries which are used throughout the paper; the trapezoidal quadrature rule is recalled in Section 3. In Section 4, we explain convergence analysis of the method.

2. Preliminaries

**Definition 1** [1]. Let \( f, g : \mathbb{R} \to \mathbb{R}_F \) be fuzzy number valued functions. The distance between \( f, g \) is defined by

\[
D^*(f, g) := \sup_{x \in \mathbb{R}} D(f(x), g(x)).
\]

**Definition 2** [5]. Let \( x : [a, b] \to \mathbb{R}_F \), the fuzzy Riemann-Liouville integral of fuzzy valued function \( x \) is defined as follows

\[
\mathcal{I}^\alpha_{a+} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)ds,
\]

for \( a \leq t \) and \( 0 < \alpha \leq 1 \).

**Definition 3** [5]. Let \( D_{gH} \in C([a, b], \mathbb{R}_F) \cap L([a, b], \mathbb{R}_F) \). The fuzzy gH-fractional Caputo differentiability of fuzzy valued function \( x \) is defined as following

\[
c^gH D^\alpha_{a+} x(t) = \mathcal{I}^{1-\alpha}_{a} (D_{gH} x(t)) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} (D_{gH} x(s))ds,
\]

where \( 0 < \alpha \leq 1 \), \( t > a \).

**Proposition 1** [3]. Let \( f : [a, b] \to \mathbb{R}_F \) be a L-Lipschitz function. Then we obtain the fuzzy trapezoidal quadrature formula

\[
D\left(\mathcal{I}^\alpha_{a} f(x)\right) = \frac{b-a}{2} \oplus [f(a) \oplus f(b)] \leq L \cdot \frac{(b-a)^2}{4}.
\]

3. Numerical approach and quadrature rule for NFFIDEs

In this section, we describe a numerical approach for the fuzzy fractional integral definition 2 which will be used for computing \( L(x, t) \) and \( f(x) \). Our method for solving Eq. (1) will be based on the following trapezoidal quadrature rule, which recalled from [3] along with a definition.

**Lemma 1** [4]. Let \( \alpha > 0 \) and \( n = [\alpha] + 1 \). Then

\[
\Gamma^{(c)} D^\alpha y(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k.
\]

According to Lemma 2.2 in [4], we convert (1) to two kinds of the following fuzzy fractional integral equations:

**Case (i):** \( y(x) = f(x) \oplus \int_0^x L(x, t) \odot G(y(t))dt, \) where \( f(x) = y(0) \oplus \int_0^x \frac{1}{\Gamma(\alpha)} (x-t)^{\alpha-1} g(t)dt, \) and

\[
L(x, t) = \frac{1}{\Gamma(\alpha)} \int_t^x (x-\tau)^{\alpha-1} k(\tau, t)d\tau.
\]

**Case (ii):** \( y(x) = g(x) \odot (-1) \int_0^x L(x, t) \odot G(y(t))dt, \) where \( g(x) = y(0) \odot (-1) \int_0^x \frac{1}{\Gamma(\alpha)} (x-t)^{\alpha-1} g(t)dt. \)

In [2], authors proved the existence and uniqueness of the solution of (1).
4. Convergence analysis

Here, firstly, we introduce a successive approximation method:

\[ y_0(x_i) = f(x_i), \]

\[ y_m(x_i) = f(x_i) \oplus \int_0^{x_i} L(x_i, t) \odot G(y_{m-1}(t))dt, \]

then, we present a numerical method based on the iterative method and trapezoidal rule to solve (1) as follows:

\[ u_0(x_i) = f(x_i), \]

\[ u_m(x_i) = f(x_i) \oplus \sum_{j=0}^{n-1} \frac{h}{2} \odot \left[ L(x_i, t) \odot G(y_{m-1}(t_j)) \oplus L(x_i, t) \odot G(y_{m-1}(t_{j+1})) \right]. \]

**Theorem.** Let \( f(x) \in C^1_L(I) = \{ f|f^{(i)} \in C_L, 0 \leq i \leq 1 \}, L(x, t) \in C^1_L(H) \) and \( C = L_1 Mb < 1 \), where \( I = [0, b] \) and \( H = \{(x, y), x \in I, y \in [0, x]\} \subset I \times I \). Let \( M = \max_{x,t \in I} |L(x, t)| \) and \( L_1 \) be Lipschitz constant of \( G \). Then

\[ D(y_m(x_i), u_m(x_i)) \leq \frac{Lb^2}{4n(1-C)}. \]

**Proof.** By using (4) and (6) for \( m = 1 \), we have

\[ D(y_1(x_i), u_1(x_i)) \leq \frac{Lb^2}{4n}. \]

Also, we have

\[ D(y_2(x_i), u_2(x_i)) \leq \frac{Lb^2}{4n} [1 + L_1 Mb]. \]

By induction, for \( m > 2 \), we get

\[ D(y_m(x_i), u_m(x_i)) \leq \frac{Lb^2}{4n} \times \frac{1}{1-C} = \frac{Lb^2}{4n(1-C)}. \]

**References**

PARAMETRIC RESONANCE OF THE SYSTEM CONSISTING OF THE CIRCULAR HOLLOW CYLINDER AND SURROUNDING ELASTIC MEDIUM UNDER ACTION IN THE INTERIOR

M.A. MEHDIYEV\textsuperscript{1,2}, S.D. AKBAROV\textsuperscript{3}

\textsuperscript{1}Institute of Mathematics and Mechanics of ANAS, Baku, Azerbaijan
\textsuperscript{2}Azerbaijan State University of Economics, Department of Mathematics, Baku, Azerbaijan
\textsuperscript{3}Yildiz Technical University, Faculty of Mechanical Engineering, Department of Mechanical Engineering, Yildiz Campus, 34349, Besiktas, Istanbul, Turkey
e-mail: akbarov@yildiz.edu.tr

In the paper [1] it was shown that under the forced vibration of the system consisting of the hollow cylinder and the surrounding elastic medium under the action time-harmonic axisymmetric ring forces on the interior of the cylinder the resonance phenomenon does not appear. In this case, the dependence between the frequency and amplitudes of the quantities characterizing the stress-strain state in the aforementioned system appearing as a result of the time-harmonic ring load has non-monotonic character. In other words, there exists such value of the frequency of the external forces under which the absolute values of the mentioned quantities have their maximum. However, in the paper [2] it was established that in the case where on the interior of the cylinder act corresponding non-axisymmetric forces, according to which, it was solved the relating three-dimensional problem the noted above dependencies have more complicated character and nevertheless the resonance phenomenon does not observe in the 3D case also. At the same time, the paper [4] establishes that if on the interior of the cylinder the axisymmetric moving constant ring load acts then under certain values moving velocity of this load the resonance type phenomenon takes place and the velocity regarding this case is called the critical velocity.

The question ”what kind of the response of the foregoing system to the time-harmonic ring forces acting on the interior of the cylinder appears in the case where these forces move with the constant velocity and this velocity is less than the corresponding critical velocity”, is the subject
of the present work. At the same time, we note that the detailed review and consideration of the works regarding the dynamics the plane layered systems have been made in the monograph [3] and in other ones listed therein.

Thus, for answering the foregoing question first we consider the mathematical modeling (or formulation) of the corresponding problem.

Consider the aforementioned “hollow cylinder + surrounding elastic medium” system the sketch of which is illustrated in Fig.1 and assume that the thickness of the wall of the cylinder is \( h \) and the external radius of the cross-section of that is \( R \). Moreover, we assume that on the inner surface of this cylinder normal time-harmonic ring forces act and these forces move along the cylinder axis with constant velocity \( V \). We associated with the central axis of the cylinder the cylindrical system of coordinates \( Or\text{ }\theta z \) and within this framework we attempt to investigate the stress-strain state in the system under consideration with utilizing the following field equations of elastodynamics.

Equations of motion:

\[
\frac{\partial \sigma_{rr}^{(k)}}{\partial r} + \frac{\partial \sigma_{rz}^{(k)}}{\partial z} + \frac{1}{r} \left( \sigma_{rr}^{(k)} - \sigma_{\theta \theta}^{(k)} \right) = \rho^{(k)} \frac{\partial^2 u_r^{(k)}}{\partial t^2}, \quad \frac{\partial \sigma_{rz}^{(k)}}{\partial r} + \frac{\partial \sigma_{zz}^{(k)}}{\partial z} + \frac{1}{r} \sigma_{rz}^{(k)} = \rho^{(k)} \frac{\partial^2 u_z^{(k)}}{\partial t^2}. \tag{1}
\]

Elasticity relations:

\[
\sigma_{nn}^{(k)} = \lambda^{(k)} \left( \varepsilon_{rr}^{(k)} + \varepsilon_{\theta \theta}^{(k)} + 2\mu^{(k)} \varepsilon_{zz}^{(k)} \right) + 2\mu^{(k)} \varepsilon_{nn}^{(k)}, \quad nn = rr; \theta \theta; zz, \quad \sigma_{rz}^{(k)} = 2\mu^{(k)} \varepsilon_{rz}^{(k)}. \tag{2}
\]

Strain – displacement relations:

\[
\varepsilon_{rr}^{(k)} = \frac{\partial u_r^{(k)}}{\partial r}, \quad \varepsilon_{\theta \theta}^{(k)} = \frac{u_\theta^{(k)}}{r}, \quad \varepsilon_{zz}^{(k)} = \frac{\partial u_z^{(k)}}{\partial z}, \quad \varepsilon_{rz}^{(k)} = \frac{1}{2} \left( \frac{\partial u_r^{(k)}}{\partial z} + \frac{\partial u_z^{(k)}}{\partial r} \right). \tag{3}
\]

In equations (1), (2) and (3) the conventional notation of the theory of elasticity is used and through the upper index \((k)\) it is indicated the belonging of the quantities to the cylinder under \( k = 2 \) and to the surrounding elastic medium under \( k = 1 \).

Consider also the formulation of the corresponding boundary and contact conditions which can be written as follows:

\[
\sigma_{rr}^{(2)} \bigg|_{r=R-h} = -P_0 \delta(z-Vt)e^{i\omega t}, \quad \sigma_{rz}^{(2)} \bigg|_{r=R-h} = 0, \tag{4}
\]

\[
\sigma_{rr}^{(1)} \bigg|_{r=R} = \sigma_{rr}^{(2)} \bigg|_{r=R}, \quad \sigma_{rz}^{(1)} \bigg|_{r=R} = \sigma_{rz}^{(2)} \bigg|_{r=R}, \quad u_r^{(1)} \bigg|_{r=R} = u_r^{(2)} \bigg|_{r=R}, \quad u_z^{(1)} \bigg|_{r=R} = u_z^{(2)} \bigg|_{r=R} \tag{5}
\]

\[
\sigma_{rr}^{(1)}; \varepsilon_{\theta \theta}^{(1)}; \varepsilon_{zz}^{(1)}; \sigma_{rz}^{(1)}; u_r^{(1)}; u_z^{(1)} \to 0, \quad \sqrt{r^2 + z^2} \to \infty. \tag{6}
\]

Thus, the investigation of the problem is reduced to the boundary-contact problem (1) – (6) for the solution to which the method developed in the papers [1, 2, 4] is employed. This method is based on the use of the moving coordinate system \( O'r'\theta'z' \) which is determined by the relations \( z' = z-Vt, \ r' = r \) and \( \theta' = \theta \). After introducing the moving coordinate system, and after representing of the sought values as \( g(r, z', t) = \tilde{g}(r, z')e^{i\omega t} \) it is employed the exponential Fourier transform

\[
\tilde{g}_F = \int_{-\infty}^{+\infty} \tilde{g}(z)e^{i\omega z'}dz'.
\]

Employing the corresponding mathematical manipulations the analytical expressions of the Fourier transforms of the sought values are determined and the originals of those are found numerically with the use of the algorithm described in the references [1-4]. In this way, it is constructed the graphs illustrated the frequency response of the amplitude of the interface normal stress acting between the constituents for various values of the velocity of the moving load. According to these graphs, it is established that there exist such values
of the moving load velocity under which resonance type phenomenon take place and on the base this reason this resonance is called as the parametric resonance. As an example for the foregoing parametric resonance, we consider the case where \( E^{(1)}/E^{(2)} = 0.05, \rho^{(1)}/\rho^{(2)} = 0.01, \nu^{(1)} = \nu^{(2)} = 0.25 \) and analyze the graphs given in Fig.2 which illustrate the frequency response of the amplitude of the dimensionless interface normal stress denoted as \( \sigma_{nn} h/P_0 \) with respect to the dimensionless frequency \( \Omega = \omega h/c_2^{(2)} \). Note that in Fig.2 the graphs constructed under \( h/R = 0.5 \) (under \( h/R = 0.2 \)) are grouped by the letter a (by the letter b).

Figure 2. The graphs of the frequency responses of the interface normal stress in the case where \( h/R = 0.5 \) (a) and \( h/R = 0.2 \)(b).

In Fig.2 the resonance frequencies are indicated through the dashed vertical lines from which follows clearly the existence of the parametric resonance and decreasing the values of these resonance frequencies with the load moving velocity \( c(=V/c_2^{(2)}) \). At the same time, in the present work, the other similar type results are presented and analyzed.

**Keywords:** Parametric resonance, dynamics of oscillating moving load, elastic medium, hollow cylinder, interface stresses, Fourier transform.

**AMS Subject Classification:** 74F10.

---

**REFERENCES**


A HYBRID ADVANCED MULTISTEP METHOD FOR SOLVING ODE

G.YU. MEHDIYEVA, V.R. IBRAHIMOV, M.N. IMANOVA

1Department of Computational Mathematics, Baku State University, Baku, Azerbaijan
e-mail: ibvag47@mail.ru

There are some classes of methods to solving initial-value problem for the ODE of the first order. It is known that each of these methods have its advantages and disadvantages. For the construction the numerical methods with the better properties, scientists have used the junction of the different methods. For example the hybrid methods are constructed one the junction of one and multistep methods. Here, for the construction the stable methods with the higher order of exactness have used the junction of the advanced and hybrid methods. Constructed the stable methods with the higher order of exactness.

Here investigate the finding of a numerical solution of the following initial value problem:

\[ y' = f(x, y), \quad y(x_0) = y_0, \quad x_0 \leq x \leq X. \]  (1)

Assume that the initial value problem (1) has a continuous unique solution defined on the segment \([x_0, X]\). To find an approximate value of the solution of the problem (1), the segment \([x_0, X]\) is divided into \(N\) equal parts by the step size \(h > 0\), and the mesh point is defined as \(x_i = x_0 + ih (i = 0, 1, ..., N)\). In addition, we denote by \(y_m\) the approximate values by the \(y(x_m)\) corresponding exact values of the solution of problem (1) at the mesh point \(x_m (m = 0, 1, ...).\)

Remark that the scientists from different countries to obtain more accurate results in solving practical problems, considered the generalization of the Euler method, in resulting which the emergence of one and multi-step methods are constructed. The known representatives of these methods are the Runge-Kutta and Adams methods, each of which has the advantages and disadvantages. Runge-Kutta methods have been fundamentally investigated by J. Butcher. The aim of this paper is to construct the methods on the junction of the advanced multistep and hybrid methods. These methods are multi-step and belong to the class of multistep methods with the constant coefficients. Multistep methods have been fundamentally investigated by Dalquist, the generalization of which is usually called the k-step methods of Obreshkov type or multistep multiderivative methods that were fundamentally investigated by Ibrahimov V.R. As is known, the advanced multistep methods are not coincide with the Adams methods and are not enter to the class of multistep Obreshkov’s methods. However, the accuracy for the known stable methods of advanced multistep type are subordinates to Dalquist’s law, which was the main obstacle in the development of the advanced multistep methods. It is known that there are stable advanced multistep methods with the order of accuracy \(p > 2[k/2] + 2\). Hybrid multistep methods, proposed by Gear and Butcher, are intensively studied in recent years. As noted, one of the popular methods for solving problem (1) is a multi-step method with the constant coefficients having the following form:

\[ \sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i}, \quad n = 0, 1, ..., N - k. \]  (2)

It is easy to understand that the method (2) is a generalization of Adams’ methods, which has been studied by many specialists from different countries. Here we assume that the coefficients of method (2) satisfy the conditions A, B and C from the work of Dahlquist. It is known that
if method (2) is stable, then \( p \leq 2[k/2] + 2 \). Here the integer valued quantity \( p \) is the degree and the integer valued quantity \( k \) is the order of the method (2). The method (2) has the degree \( p \) if the following is holds:

\[
\sum_{i=0}^{k} (\alpha_i y(x + ih) - h \beta_i y'(x + ih)) = O(h^{p+1}), \quad h \to 0.
\] (3)

And the method obtained from (2) is said to be stable if the roots of the polynomial

\[
\rho(\lambda) \equiv \alpha_k \lambda^k + \alpha_{k-1} \lambda^{k-1} + \ldots + \alpha_1 \lambda + \alpha_0
\]

are lies inside of the unit circle, on the boundary of which there are no multiple roots.

As can be seen from the Daliaquist barrier, the maximum accuracy of the stable method obtained from (2) almost coincides with the number of mesh points used in its construction. Therefore the specialists used the Richardson extrapolation, the Hamming methods, and linear combination of some methods for the construction of stable multistep methods with a high degree. And some authors suggested using the following method for constructing a more accurate method:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i g_{n+i},
\]

where the function \( g(x, y) \) is defined as:

\[
g(x, y) = f_x(x, y) + f_y(x, y) f(x, y).
\]

For the construction of the more accurate stable multistep methods, the following advanced multistep method is proposed:

\[
\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i}, \quad (m > 0; \ \alpha_{k-m} \neq 0).
\]

(5)

Some scientists suggested using the following formulas to increase the exactness of the stable method obtained from formula (2):

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{j=0}^{r} h^j \sum_{i=0}^{k} \beta_{i,j} y^{(j)} \quad (\alpha_k \neq 0),
\]

(6)

which are fundamentally investigated by V.R.Ibrahimov. We note that when the method (4) is applied to solving of problem (1), the amount of computational work is tweaks. However, method (4) can be applied to solving of the initial value problem for ordinary differential equations with the special structure. In this case the method (4) is the generalizes the well-known Stormer method. We are considering that it is desirable to apply the method (6) to solving of the following problem:

\[
y^{(r)} = F(x, y, y', ..., y^{(r-1)}); \quad y^{(j)}(x_0) = y^{(j)}_0 \quad (j = 0, 1, ..., r - 1).
\]

(7)

Note that for the application of the method to solving of the problem (7), the definitions of explicit and implicit methods may differ from the known concepts. For example, method (6) is considered explicit if \( \beta_{k,j} = 0 \) \((j = 0, 1, ..., r)\). However, the method obtained from formula (6) can also be explicit in the case when \( \beta_{k,r} = 0 \). As it was noted, for the construction of more precise methods, it is possible to use the hybrid methods, which in more general form can be written as the following:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \sum_{i=0}^{k} \gamma_i f_{n+i+v_i} \quad (|\nu_i| < 1; \ i = 0, 1, ..., k).
\]

(8)

This method has been investigated by many authors. Thus, we have obtained two directions for constructing more precise methods, consisting in using methods of type (4) and type (5).
To construct effective methods, here is proposed to use the methods obtained from formulas (5) and (8). Thus, for the construction of hybrid advanced multistep methods is proposed to use the following formula:

\[ \sum_{i=0}^{k-m} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \sum_{i=0}^{k} \gamma_i f_{n+i+\nu_i} \quad (m > 0; \ |\nu_i| < 1; \ i = 0, 1, ..., k). \tag{9} \]

Method (9) was constructed taking into account the fact that in the class of the methods (5), there exist stable methods with the degree \( p = k + m + 1 \) \( (k \geq 3m) \), and there exist stable methods with \( p = 2k + 2 \) in the class of methods of type (8). Therefore, method (9) was constructed at the junction of the advanced multistep and hybrid methods. Here we try to choose the quantities \( \alpha_i \) \( (i = 0, 1, ..., k - m) \), \( \beta_i, \gamma_i; \nu_i (i = 0, 1, ..., k) \) so that the methods obtained from formula (9) will be have high accuracy and the extended stability region. In order to study the methods (9), we assume that the coefficients \( \alpha_i \) \( (i = 0, 1, ..., k - m) \), \( \beta_i, \gamma_i \) \( (i = 0, 1, ..., k) \) satisfies the following conditions:

A: The coefficients \( \alpha_i (i = 0, 1, 2, ..., k - m) \), \( \beta_i, \gamma_i, \nu_i (i = 0, 1, 2, ..., k) \) are some real numbers, moreover, \( \alpha_k \neq 0 \).

B: Characteristic polynomials

\[ \rho(\lambda) \equiv \sum_{i=0}^{k-m} \alpha_i \lambda^i, \sigma(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i; \gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^{i+\nu_i}. \]

have no common multiply factor different from the constant. C: \( \sigma(1) + \gamma(1) \neq 0 \) and \( p \geq 1 \).

For the purpose of constructing a system of algebraic equations, to find the coefficients of the method (9) \( \alpha_i (i = 0, 1, ..., k - m) \), \( \beta_i, \gamma_i, \nu_i (i = 0, 1, ..., k) \), we use the method of undetermined coefficients, in the results of which receive a system of nonlinear algebraic equations, which for \( m = 1 \) and \( k = 2 \)can be written as the following:

\[ 2\beta_2 + \beta_1 + \beta_0 + \gamma_2 + \gamma_1 + \gamma_0 = 1, \]

\[ 2\beta_2 + \beta_1 + \beta_0 + \gamma_2 + \gamma_1 + \gamma_0 = 1/(j+1), \quad (j = 1, 2, ..., 9) \tag{10} \]

To simplify the system (10), let us consider the case \( \nu_1 = 0; \nu_0 = 1/2; \nu_1 = 0; \nu_2 = -1/2; \) and \( \gamma_1 = \beta_1 \). Then, by solving the system (10) we obtain that.

\( \beta_0 = 29/180; \beta_1 = 6/90; \beta_2 = -1/180; \gamma_0 = 21/45; \gamma_1 = 6/90; \gamma_2 = 1/45 \)

By using the obtained solution in the formula (9), one can obtain the following hybrid advanced multistep method:

\[ y_{n+1} = y_n + h(29f_n + 24f_{n+1} - f_{n+2})/180 + h(62f_{n+1/2} + 2f_{n+3/2})/90. \tag{11} \]

This method is stable and has the degree \( p = 5 \).

Keywords: Advanced multistep methods, initial value problem, stability, the order and degree.

AMS Subject Classification: 65L05.

References


METHOD OF THE RESERVE RESOURCES DETERMINATION FOR THE DISTRIBUTED COMPUTER SYSTEMS WITH THE NETWORK-CENTRIC RESOURCE CONTROL

V.YE. MUKHIN\(^1\), YA.I. KORNAGA\(^1\), S.H. ABDULLAYEV\(^2\), O.YU. GERASYMENKO\(^3\), YU.A. BAZAKA\(^1\)

\(^1\)National Technical University of Ukraine "Igor Sikorsky Kiev Polytechnic Institute", Ukraine, Kiev, Pr. Pobedy, 37
\(^2\)Institute of Information Technology, Azerbaijan, Baku, Str. B. Vahabzade, 9A
\(^3\)Taras Shevchenko National University of Kiev, Ukraine, Kiev, Vladimirskaya Street, 60
e-mail: v.mukhin@kpi.ua

1. INTRODUCTION

Nowadays, there are several approaches to Distributed Computer Systems (DCS) resources control, the main of which are centralized, decentralized and hierarchical approaches. Taking into account the scalability of DCS, now it is particularly important the resource control mechanisms based on the decentralized approach. The decentralized approach has the following advantages: fault tolerance, the flexible scalability, high availability. However, the implementation of this approach is a very difficult task, because in this case in the system there is no any central element, which would be store all the information about system resources and performs the resources control. In order to improve the DCS resources control on the decentralized approach it was proposed to implement the module of Resource Management System (RMS) in DCS, the functions of which is based on network-centric control principles [1].

2. THE PROBLEM STATEMENT

In asynchronous mode for the tasks with a high level of parallelization module in RMS module, which is based on the network-centric approach, there was introduced the resources reserve ratio, otherwise for intensive flow of tasks all the DCS resources are distributed between the tasks, and the average time of task execution is significantly increased. We assume that the the parameters of the DCS functioning is significantly dependent on the reserve ratio. It is important to develop a mechanism for reserve ratio control to improve the efficiency of DCS functioning. Since the high reserve ratio will increase DCS resources idle time, it is wisely to take the rate of DCS resource utilization and the average time spent in queuing as parameters which are determining the effectiveness of the DCS functioning.

3. DEVELOPMENT OF A METHOD FOR DETERMINING THE REQUIRED NUMBER OF DCS RESOURCES

Before to get to the development of the method, we consider a tasks model in a parallel form, which is used in this study. The task is presented in MPL-form (multi-layer parallel form). Let the task consist of five levels, on each of which there is a certain number of sub-tasks (subtask, st) of the task.
In [1] the basic concept of DCS resource control based on the network-centric approach is presented. At asynchronous mode for the tasks with a high level of parallelization in the RMS module was introduced the resource reserve ratio, since otherwise for the intensive tasks flow all resources are distributed among the tasks and the average time of one task execution is increased significantly. It is necessary to in order to improve the DCS functioning efficiency.

In order to develop a mechanism for control the number of reserved resources for a distributed system we use the neural network because they are among the most common mechanisms for the image recognition, and in this case they are used for the recognition of the situation, or in other words, the assessment of the situation.

The number of reserved resources is affected by both the task parameters and the DCS parameters, in particular, the performance of the compute nodes. It is desirable to determine the amount of reserve resources for each task separately. The following steps should be performed:
- to formalize the task structure and to define a set of tasks that will be used to learn the neural network;
- to perform simulation of the DCS functioning with RMS module, based on network-centric approach in order to collect data for the neural network learning, with a set of input tasks selected in the previous step, and the tasks in the set must have a different structure;
- using a neural network to determine the reserve ratio for each next task that will arrive in the DCS, and apply the received ratio to the RMS along with the task.

The parameters of the task that must be submitted to the inputs of the neural network should fully reflect the task structure, so the following parameters are used: the number of levels, the number of subtasks, the average computational complexity of the subtask, the average number of input data streams per subtask (arcsOnStAvg), the level number with the maximum number subtasks (levMaxVol), the maximum number of subtasks at all levels (maxLevVol), the average number of subtasks at the level (stOnLevAvg), the interaction coefficient between the subtasks (interconCoef).

4. Experiments to study the method of determination of the required amount of reserve resources for a task in an asynchronous mode

For the experiment, the it was used the RMS module on the network-centric approach, which was slightly modified. The modification concerned the way to determine the amount of reserve resources for each task [2]. In the original version, the reserve ratio was set with a value of 0.35, and the ration was used to calculate the amount of reserve resources, i.e. the maximum number of subtasks at the level was multiplied by this ratio. During the experiment, the amount of reserve resources for the task was determined by the data which were obtained as a result of the task parameters was inputed to the neural network.

So, for testing of the method there were generated 5 sets of tasks, and for each of the tasks for all sets the input data for the neural network was generated, next they were fed to the inputs of the neural network and the predicted number of requests for additional DCS resources from the task was obtained. Based on the task parameters and on the number of DCS resource requests, determined by the neural network, the number of reserve resources required for the task was calculated, which, as one of the task parameters, was fed to the input of the simulating system.

For each set of tasks, there were conducted 8 experiments for two different methods of resources reserve ratio determining (with 0.35 reservation ratio and with the proposed method). During the experiments, for each task the waiting time in the queue and the execution time on the DCS resources were recorded. According to the received data, the average waiting time of the task in the queue and the average time of the task execution fixed in quanta were determined.
5. ANALYSIS OF EXPERIMENTAL RESULTS ON THE METHOD OF DETERMINATION OF THE REQUIRED AMOUNT OF RESERVE RESOURCES FOR A TASK IN AN ASYNCHRONOUS MODE

An analysis of the experimental results showed that the average waiting time for the task in the queue was decreased with the application of the suggested method, although the average time for the task was somewhat increased. This is explained by the fact that DCS resources with less computing power have been used more actively, and, consequently, tasks began to be executed longer. However, these time losses completely cover the reduction in the waiting time of the task in the queue, i.e. the total time of the task stay in the DCS also was decreased, which is shown in Fig. 1.

Thus, the decrease in the average waiting time of the task in the queue is from 13.95% to 28.29%, but due to the increase in the average time of the task, which is from 2.15% to 4.81%, the average time of the task stay in the DCS is decreased by at least 8.15% for the third set of tasks and a maximum of 15.25% for the fifth set of tasks.

![Figure 1. Comparison of the average waiting time of the task in the queue and the average time of the task execution in relation to the average time spent by the task in the DCS for task sets and depending on the method of determining the amount of the resources reserve for the task](image)

6. CONCLUSIONS AND FURTHER RESEARCH

In this paper is suggested a method for determining the amount of reserve resources for tasks in an asynchronous mode in a distributed computer system with network-centric control. This method is based on the mechanism of neural networks.

The experimental studies have shown the effectiveness of this method, although it should be noted that when using this method it is necessary that the tasks that are put into execution in the DCS should have parameters similar to the parameters of the task set (the number of levels, the number of subtasks at the level, etc.) which were used to learn the neural network. Otherwise, it is necessary to form a new set of data for learning of the neural network and re-learn the neural network on this set.

Keywords: Network-centric approach, resource control, reservation.

AMS Subject Classification: 93C62.

REFERENCES


CRUDE OIL PRICE FORECASTING TECHNIQUES IN THE WORLD MARKET

ADALAT MURADOV, YADULLA HASANLI, NAZIM HAJIYEV

1 Azerbaijan State University of Economics (UNEC), Baku, Azerbaijan
2 Scientific Research Institute of Economic Studies, Azerbaijan State University of Economics (UNEC), Baku, Azerbaijan
3 Institute of Control Systems of ANAS, Baku, Azerbaijan
4 Davis Center for Russian and Eurasian Studies, Harvard University, Boston, USA

E-mail: adalet_muradov@yahoo.com, yadulla_hasanli@yahoo.com, nazimxx@yahoo.com

Abstract. The existing forecasting methods is studied in the research by considering all these cases and forecasting of mean oil price in the world market for medium and long-term period with Trend, ARIMA, Holt, ARCH / GARCH models have been forecasted for the 2018-2020.

Keywords: Crude oil prices, forecasting methods, ARIMA, Holt, ARCH /GARCH models.

AMS Subject Classification: 62P20, 91B76.

1. Introduction

It is an unavoidable fact that oil prices like other energy resources play an important role in the development tendency of the world. However, no exact method is known in the science to accurately forecast oil prices. The experience indicates that information possessed by international institutions for an accurate forecast of crude oil prices does not reflect the reality. However, researches are being conducted towards this direction. The challenge in terms of oil price forecasting is associated with political factors together with economic factors. As mentioned earlier, oil price forecasting is performed by international institutions. However, prices which are forecasted by these institutions do usually not reflect the reality. These diversions get more critical during an economic crisis and with political ambitions.

2. Methodology

International institutions use econometric models together with fuzzy and artificial neuron expert assessment systems which are characterized with their availability and accuracy by for forecasting crude oil prices. However, there is not an accepted idea which method should be used. Quantitative and qualitative methods exist for oil price forecasting. Quantitative variables which affect oil prices are employed through quantitative methods for crude oil price forecasting. This method is divided into econometric and non-standard techniques.

Econometric models include the followings: 1.Time series models; 2.Financial models; 3.Structural models.

The most applied non-standard models for crude oil price forecasting are artificial neuron systems [2] and support vector machines [1].

A qualitative method assesses the impact of wars and natural catastrophes on the price of oil. These methods have been increasingly used for oil price forecasting in the recent academic
literature. It is important to note that many qualitative methods exist for oil price forecasting. These models include methods like Delphi method, belief networks, fuzzy logic, and expert systems as well as web text mining. Time series models are used for future oil price forecasting based on historical information. These models are usually used under the following cases: (1) If the provided data (statistical indicators) possesses a systematic pattern like autocorrelation; (2) If there are many explanatory variables and their mutual relationship requires the establishment of a more complicated model; (3) If explanatory variables forecasting is required for forecasting dependent variable (dependent variable forecasting takes much time in terms of explanatory variables forecasting).

Time series models are divided into three main categories:
1. Simple models; 2. Exponential smoothing models; 3. Autoregressive models (ARIMA); 4. Holt model; 5. ARCH/GARCH models.

In general, structural models for oil price forecasting are classified as follows:
1. OPEC behavioral models; 2. Reserves model; 3. Combination of OPEC behavioral and reserve models; 4. Demand and supply models; 5. Non-oil models.

There are a number of forecasting methods. Econometric forecasting is the most widely applied one among them in practice. The world oil market price is affected by multiple factors [3]. Among these factors, there are political factors along with economic factors. Assessment of impact of multiple factors on oil prices is more effective with structural modeling. However, use of this type of models in forecasting makes some difficulties. The difficulties are due to the fact that forecasts of the factors affecting the oil price is also needed. On the other hand, there are affective quality factors that not only availability of their forecasts, but also presence of the statistics of the retrospective time is problematic. Thus, the error in forecasting of any of the affective factors may lead to greater error in the outcome indicator. The existing forecasting methods is studied in the research by considering all these cases and forecasting of mean oil price in the world market for medium and long-term period with Trend, ARIMA, Holt, ARCH/GARCH models have been preferred. The following indicators in the existing models that valid for forecasting are compared: Root Mean Squared Error and Mean Absolute Error.

Considering the continuous changes in the world, building of short-term and retrospective forecasts can be considered reasonable. In the study, forecast of the world oil market price was made by means of the trend and ARIMA (Auto Regressive Integrated Moving Average) models. It should be noted that initial forecasting of the average oil price, including the price of AzerLight oil in the world market by means of Trend, ARIMA models has proved itself and deviated from the factual values more than the forecasts of the international organizations [4].

An econometric model was developed based on the annual data of 1975-2017.

The econometric model was obtained as follows.

\[
AOP_{MEAN} = -15.8113291652 + 1.36381483826 \times \text{TREND} - 27.4644306826 \times \text{DUMMY2008} - 27.1897200003 \times \text{DUMMY2015}
\]

(Std. Error) (8.410049) (0.236180) (4.703029) (4.590126).

R-squared=0.960661; Adjusted R-squared=0.954105; Durbin-Watson stat=2.116437; Prob(F-statistic)= 0.000000.

3. Results

It should be noted that for the purpose of elimination of autoregression and moving average during the current period, the AR factors of set 1 and 9, and MA of set 10 were included.

Where, AOP_MEAN – Brent and WTI average oil price, DUMMY2008 and DUMMY2015 are artificial variations and characterize the sharp decline of oil prices due to the crisis occurred in 2008 and 2015 respectively.

The statistical characteristics of (1) and relevant tests have shown that the model is adequate. The R-squared and the adjusted R-squared are significantly closer to each other and to the unit.
F-statistic = 0.000000 shows that the quality of the R-squared is high. Durbin-Watson statistic are significantly close to 2 due to absence of autocorrelation of the first set of balances. The Actual and Fitted prices of oil are very close. All of these factors show that application of the model (1) is useful for forecasting purposes. So, according to the model (1), the forecast of the average Brent and WTI oil price is provided in the Table 1 below under the model (1).

Table 1. Forecast of average Brent and WTI oil price (AQP\_MEAN), 1 Dollar-barrel.

<table>
<thead>
<tr>
<th>year</th>
<th>Price forecast</th>
</tr>
</thead>
<tbody>
<tr>
<td>2018</td>
<td>63.3</td>
</tr>
<tr>
<td>2019</td>
<td>64.7</td>
</tr>
<tr>
<td>2020</td>
<td>66.0</td>
</tr>
</tbody>
</table>

Source: Authors’ calculations.

As the prices of Azerbaijan oil, AzerLight in the world market are closely dependent on other oil prices, its forecast prices are forecast by building of the regression models depend and on the prices of other oils. It should be noted that the price of AzerLight in the world markets is slightly higher than those of other oils. Considering the trends of the recent years, the following logarithmic-linear regression model has been evaluated in order to forecast the price of AzerLight oil in the world market.

\[
\log(\text{AzerLight}) = 0.242518007106 + 0.951590281159 \times \log(\text{AOP\_MEAN})
\]

R-squared=0.987049; Adjusted R-squared=0.982732; Prob(F-statistic)= 0.000628

The model shows that the elasticity ratio of the AzerLight oil compared to the AOP\_MEAN factor is 0.951590281159. That means that the rise of 1 percent in the average price of Brent and WTI oil in the world markets leads to increase of AzerLight oil price about 0.95% per barrel. Thus, according to the model (2), the forecast prices for AzerLight oil for 2018-2020 are provided as follows (1 dollar/barrel) under the table below.

Table 2. Forecast price of AzerLight oil in the world market (1 dollar-barrel).

<table>
<thead>
<tr>
<th>year</th>
<th>min.</th>
<th>mid</th>
<th>Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2018</td>
<td>52.1</td>
<td>65.7</td>
<td>81.2</td>
</tr>
<tr>
<td>2019</td>
<td>57.3</td>
<td>73</td>
<td>87.1</td>
</tr>
<tr>
<td>2020</td>
<td>59.6</td>
<td>75.4</td>
<td>88.3</td>
</tr>
</tbody>
</table>

Source: Authors’ calculations.

For comparison, the international Organizations’ forecast given on the world oil market price is provided in the table below (1 dollar/barrel) [3].

Table 3. Actual and forecast price on the world oil market (1 dollar/barrel).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>WorldBank</td>
<td>48.3</td>
<td>43</td>
<td>55.9</td>
<td>53</td>
<td>57.6</td>
<td>59.5</td>
</tr>
<tr>
<td>US Energy Administration</td>
<td>39.3</td>
<td>43</td>
<td>48.5</td>
<td>53</td>
<td>51.2</td>
<td>54.5</td>
</tr>
<tr>
<td>UNEC (AzerLight)</td>
<td>51.5</td>
<td>47</td>
<td>59.0</td>
<td>55</td>
<td>65.7</td>
<td>73</td>
</tr>
</tbody>
</table>

Source: Authors’ calculations.

Source: https://www.eia.gov/outlooks/aeo/tables_ref.php

REFERENCES

BOUNDARY VALUE PROBLEM FOR THE ANISOTROPIC TYPE
CONVOLUTION EQUATIONS

HUMMET K. MUSAEV

1Baku State University, Department of Differential and Integral Equation, Baku, Azerbaijan
e-mail: gkm55@mail.ru

The coercive properties of convolution-differential equations are investigated. In this paper by using Fourier multiplier theorems are find sufficient conditions that guarantee the coercitivity of these problem under consideration in weighted \( L^p \)-spaces with mixed norm. On the other hand thanks to \( R^- \) positivity and boundedness of the corresponding operator these results are applied to obtain the boundary value problems for convolution-differential equations.

Elliptic boundary value problems have been investigated systematically beginning to the second half of the last century. Boundary value problems for differential-operator equations have been studied in [1-3].

In recent years, maximal regularity properties for differential-operator equations, especially parabolic and elliptic type have been studied extensively e.g in [1, 4] and the references therein. In this notes we study the boundary value problems for the anisotropic type convolution equations. The main aim of the present paper is to study the following bondary value problem (BVP)

\[
\sum_{|\alpha| \leq l} a_\alpha * D^\alpha u + \sum_{|\alpha| \leq 2m} b_\alpha c_\alpha D^\alpha_p * u = f(x, y), \\
B_j u = \sum_{|\beta| \leq m_j} b_j\beta(y) D^\beta_y u = 0,
\]

for all \( f \in L^p(\Omega) \) \( j = 1, m \), where \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \), \( u = u(x, y) \), \( y = (y_1, ..., y_\mu) \), and \( \alpha_\alpha = \alpha_\alpha(x) \), \( b_\alpha = b_\alpha(x) \), \( c_\alpha = c_\alpha(y) \) are complex valued functions.

First we consider the convolution differential operator equation

\[
\sum_{|\alpha| \leq l} a_\alpha * D^\alpha u + A_\lambda * u = f(x)
\]

in \( L^p(\mathbb{R}^n; E) \), where \( A_\lambda = A + \lambda \), \( A = A(x) \) is a possible unbounded operator in a Banach space \( E \), \( a_\alpha = a_\alpha(x) \) are complex valued functions on \( \mathbb{R}^n \).

In [3] under certain conditions, it was proved that for equation (2) there exists a unique solution and the following coercive uniform estimate holds

\[
\sum_{|\alpha| \leq l} |\lambda|^{|\alpha|} ||a_\alpha * D^\alpha u||_{L^p(\mathbb{R}^n; E)} + ||A * u||_{L^p(\mathbb{R}^n; E)} + |\lambda||u||_{L^p(\mathbb{R}^n; E)} \leq C ||f||_{L^p(\mathbb{R}^n; E)}
\]

for all \( f \in L^p(\mathbb{R}^n; E) \) and \( \lambda \in S_\varphi \), \( p \in (1, \infty) \), \( \varphi \in [0, \pi) \).

In this paper we established the maximal regularity properties of the problem (1). Main tools of this section are the operator–valued Fourier multipliers. At the same time, coercive estimate
for (1) is derived by using the representation formula for solution of problem (1) and operator valued multiplier results in weighted mixed norms space \( L_{p,\gamma}(\Omega) \).

\( L_{p,\gamma}(\Omega) \) will denote the space of all \( p \)-summable scalar-valued functions with mixed norm, i.e. the space of all measurable functions \( f \) defined on \( \Omega \), for which

\[
\| f \|_{L_{p,\gamma}(\Omega)} = \left( \frac{1}{p} \int_\Omega \left( \int_\Omega |f(x,y)|^{pm} \gamma(x) \, dx \right)^{\frac{1}{pm}} \, dy \right)^{\frac{1}{p}} < \infty,
\]

where \( \Omega = R^n \times \Omega, p = (p_1, p) \) and \( \gamma(x) = |x|^\alpha \), \( \Omega \subset R^m \) is an open connected set with compact \( C^{2m} \)-boundary \( \partial \Omega \). Consider the BVP for integro-differential equation

\[
(L + \lambda) u = \sum_{|\alpha| \leq \ell} \varepsilon_\alpha a_\alpha \ast D^\alpha u + \sum_{|\alpha| \leq 2m} \left( b_\alpha c_\alpha D^\alpha_y u + \lambda \right) \ast u
\]

\[
= f(x, y), \quad x \in R^n, \quad y \in \Omega \subset R^m,
\]

\[
B_j u = \sum_{|\beta| \leq m_j} b_{j\beta}(y) D^\beta_x u(x, y) = 0, \quad y \in \partial \Omega, \quad j = 1, 2, \ldots, m,
\]

where

\[
\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \quad a_\alpha = a_\alpha(x), \quad u = u(x, y),
\]

\[
D_j = -i \frac{\partial}{\partial y_j}, \quad y = (y_1, \ldots, y_m), \quad b_\alpha = b_\alpha(x), \quad c_\alpha = c_\alpha(y).
\]

We present the following main result:

**Theorem 1.** Let the following conditions be satisfied:

(1) for \( \xi \in R^n \), \( L(\xi) = \sum_{|\alpha| \leq \ell} \hat{a}_\alpha(\xi)(i\xi)^\alpha \in S_{\varphi_1}, \varphi_1 \in [0, \pi), \| L(\xi) \| \geq C \sum_{k=1}^{n} \| \hat{a}_k \| \| \xi_k \| ^{\hat{\ell}}, \hat{a}_\alpha, \hat{b}_\alpha \in C^{(n)}(R^n) \) and there are positive constants \( C_1 \) and \( C_2 \) so that

\[
|\xi|^{|eta|} \left| D^\beta \hat{a}_\alpha(\xi) \right| \leq C_1, \quad |\xi|^{|eta|} \left| D^\beta \hat{b}_\alpha(\xi) \right| \leq C_2 \left| \hat{b}_\alpha(\xi) \right|,
\]

\( \xi \in R^n \setminus \{0\}, \beta_k \in \{0, 1\}, 0 \leq |\beta| \leq n; \)

(2) \( c_\alpha \in C(\hat{\Omega}) \) for each \( |\alpha| = 2m \) and \( c_\alpha \in L_\infty(\Omega) + L_{r_k}(\Omega) \) for each \( |\alpha| = k < 2m \) with \( r_k \geq p_1, 2m - k > \frac{1}{r_k} \);

(3) \( b_{j\beta} \in C^{2m-m_j}(\partial \Omega) \) for each \( j, \beta, m_j < 2m; \)

(4) for \( y \in \hat{\Omega}, \xi \in R^m, \sigma \in S_{\varphi_0}, \varphi_0 \in (0, \frac{\pi}{2}), |\xi| + |\sigma| \neq 0 \) let \( \sigma + \sum_{|\alpha| = 2m} c_\alpha(y) \xi_\alpha \neq 0; \)

(5) for each \( y_0 \in \partial \Omega \) local BVP in local coordinates corresponding to \( y_0 \)

\[
\sigma + \sum_{|\alpha| = 2m} c_\alpha(y_0) D^\alpha \vartheta(y) = 0,
\]

\[
B_{j_0} \vartheta = \sum_{|\beta| = m_j} b_{j_\beta}(y_0) D^\beta \vartheta(y) = h_j, \quad j = 1, 2, \ldots, m
\]

has a unique solution \( \vartheta \in C_0(R_+) \) for all \( h = (h_1, h_2, \ldots, h_m) \in R^m \) and for \( \xi' \in R^{m-1} \) with \( |\xi'| + |\lambda| \neq 0; \)
Then, for $f \in L_{p,\gamma}(\Omega)$ and $\lambda \in S_\varphi$ the problem (1.3) - (1.4) has a unique solution $u \in W_{p,\gamma}^l(\Omega)$ and the following coercive uniform estimate holds

$$
\sum_{|\alpha| \leq l} |\lambda|^{1-|\alpha|/l} \|a_\alpha * D^\alpha u\|_{L_{p,\gamma}(\Omega)} + \|\lambda| u\|_{L_{p,\gamma}(\Omega)}
+ \sum_{|\alpha| \leq 2m} |b_\alpha c_\alpha D^\alpha u\|_{L_{p,\gamma}(\Omega)} \leq C \|f\|_{L_{p,\gamma}(\Omega)}.
$$

**Keywords:** Boundary value problems, $R$-positivity, convolution differential-operator equation, weighted spaces, $p-$summable scalar-valued functions.

**AMS Subject Classification:** 34G10, 45J05.

**References**


NECESSARY CONDITIONS FOR THE FREDHOLMNESS OF
THREE-DIMENSIONAL HELMHOLTZ EQUATION WITH NONLOCAL
BOUNDARY VALUE CONDITIONS

Y.Y. MUSTAFAYEVA\textsuperscript{1}, N.A. ALIYEV\textsuperscript{1}

\textsuperscript{1}Baku State University, Baku, Azerbaijan
e-mail: helenmust@bsu.edu.az

The paper is dedicated to the investigation of the Fredholm property of boundary value problems with nonlocal boundary conditions for a three-dimensional Helmholtz equation.

As is known, for an ordinary differential equation the number of additional conditions (Cauchy conditions or boundary conditions) always coincides with the order of the equation in question.

For nonlocal boundary value problems the authors have found the possibility of proving Fredholm property with the help of so-called necessary conditions. It should be noted that for an ordinary differential equation these necessary conditions similar to nonlocal boundary conditions are mentioned by A.A.Dezin [1-3].

Let us consider the three-dimensional Helmholtz equation in a convex in the direction of $x_3$ domain $D \subset \mathbb{R}^3$ whose projection onto plane $Ox_1x_2 = Ox'$ is domain $S \subset Ox_1x_2$, $\Gamma$ is the boundary (surface) of the domain $D$:

$$\Delta u + a^2 u(x) = -f(x), \ x = (x_1, x_2, x_3) \in D \subset \mathbb{R}^3$$

with nonlocal boundary conditions

$$l_iu = \sum_{j=1}^{3} \left[ \alpha_{ij}^{(1)}(x') \frac{\partial u(x)}{\partial x_j} |_{x_3=\gamma_1(x')} + \alpha_{ij}^{(2)}(x') \frac{\partial u(x)}{\partial x_j} |_{x_3=\gamma_2(x')} \right] +$$

$$+ \sum_{k=1}^{2} \alpha_{ij}^{(k)}(x') u(x', \gamma_k(x')) = 0, \ i = 1, 2; \ x' \in S,$$

and additional Dirichlet's condition on the equator $L$ of the surface $\Gamma$

$$u(x) = f_0(x), \ x \in L = \Gamma_1 \bigcap \Gamma_2.$$  

where $S$ is the projection of the domain $D$ onto the plane $Ox_1x_2 = Ox'$, $\Gamma = \partial D$ is Lyapunov's surface; $L$ is the equator connecting the upper and lower semi-surfaces $\Gamma_1$ and $\Gamma_2$: $\Gamma_k = \{ \xi = (\xi_1, \xi_2, \xi_3) : \xi_3 = \gamma_k(\xi'), \xi' = (\xi_1, \xi_2) \in S \}, k = 1, 2$, where $\xi_3 = \gamma_k(\xi_1, \xi_2), k = 1, 2$, are the equations of the semi-surfaces $\Gamma_1$ and $\Gamma_2$ (the convexity of the domain $D$ in the direction of $Ox_3$ provides the existence of such equations), functions $\gamma_k(\xi'), k = 1, 2$, are twice differentiable with respect to the both of variables $\xi_1, \xi_2$; the coefficients $\alpha_{ij}^{(k)}(x')$ satisfy H"older condition in $S$; $\alpha_{ij}^{(k)}(x'), i, k = 1, 2, f(x)$ and $f_0(x)$ are continuous functions. The fundamental solution for the three-dimensional Helmholtz operator $(\Delta + a^2 I)U(x) = \delta(x))$ has the form of

$$U(x) = -\varepsilon^{-ia|x|} e^{-ia|x|} \frac{e^{ia|x-\xi|}}{4\pi |x-\xi|} [4],$$

from which we choose the following

$$U(x - \xi) = -\frac{e^{ia|x-\xi|}}{4\pi |x-\xi|}.$$
The necessary conditions of the solvability of boundary value problem (1)-(3) for three-dimensional Helmholtz equation are obtained. The first necessary condition has the form:

$$\frac{1}{2} u(\xi) = - \int_{\Gamma} \left( \frac{\partial u(x)}{\partial \nu_x} U(x - \xi) - u(x) \frac{\partial U(x - \xi)}{\partial \nu_x} \right) dx + \int_{D} f(x) U(x - \xi) dx, \quad \xi \in \Gamma. \quad (5)$$

Thus we have proved

**Theorem 1.** Let a convex along the direction $x_3$ domain $D \subset R^3$ be bounded with the boundary $\Gamma$ which is a Lyapunov surface. Then the obtained first necessary condition (5) is regular. The rest 2nd, 3rd and 4th necessary conditions are obtained in the form:

$$\frac{1}{2} \frac{\partial u}{\partial \xi_1} \Big|_{\xi_3 = \gamma_k(\xi')} = (-1)^{k+1} \int_{S} \frac{\partial u(x)}{\partial x_m} \Big|_{x_3 = \gamma_k(x')} \frac{1}{4\pi |x' - \xi'|^2} K_{mn}^{(k)}(x', \xi') \frac{dx'}{P_k(x', \xi')} \cos(\nu_x, x_3) + \ldots, \quad (6)$$

where three dots designate the sum of nonsingular terms, taking into account the designations:

$$K_{ij}(x, \xi) = (\cos(x - \xi, x_i) \cos(\nu_x, x_j) - \cos(x - \xi, x_j) \cos(\nu_x, x_i)) \times e^{ia|x - \xi|} (1 - ia|x - \xi|) \quad (7)$$

$$K_{ij}^{(k)}(x', \xi') = K_{ij}(x, \xi) \bigg|_{x_3 = \gamma_k(x'), \xi_3 = \gamma_k(\xi')}, \quad k = 1, 2, \quad (8)$$

and

$$P_k(x', \xi') = 1 + \sum_{m=1}^{2} \left( \frac{\partial \gamma_k(x')}{\partial x_m} \right)^2 \cos^2(x' - \xi', x_m) + 2 \frac{\partial \gamma_k(x')}{\partial x_1} \frac{\partial \gamma_k(x')}{\partial x_2} \cos(x' - \xi', x_1) \cos(x' - \xi', x_2) + O(|x' - \xi'|).$$

Introducing a new designation

$$Q_k(x', \xi') = \left( \cos(x - \xi, \nu_x) e^{ia|x - \xi|} (1 - ia|x - \xi|) \right) \bigg|_{\xi_3 = \gamma_k(\xi'), x_3 = \gamma_k(x')}$$

and taking into account the above designations we obtain the 1st necessary condition in the form of (for $k=1, 2$):

$$\frac{1}{2} u(\xi) \Big|_{\xi_3 = \gamma_k(\xi')} = (-1)^k \frac{1}{4\pi} \int_{S} u(x) \Big|_{x_3 = \gamma_k(x')} \times \frac{Q_k(x', \xi')}{P_k(x', \xi')} \frac{dx'}{|x' - \xi'|^2 \cos(\nu_x, x_3)} + \ldots \quad (9)$$

**Theorem 2.** Under assumptions of Theorem 1 necessary conditions (6) are singular. To regularize the singular necessary conditions we build linear combinations

$$\sum_{j=1}^{3} \left( \beta_{ij}^{(1)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \bigg|_{\xi_3 = \gamma_1(\xi')} + \beta_{ij}^{(2)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \bigg|_{\xi_3 = \gamma_2(\xi')} \right)$$

...
where the functions $\beta_{ij}^{(k)}(\xi')$ satisfy Hölder condition. Then we use here the obtained necessary conditions. Hence we get a system of 6 equations for each $i=1,2$: 

$$
(-1)^{k} \beta_{ii}^{(k)}(x') \frac{K_{ij}^{(k)}(x', x')}{P_{ki}(x', x')} + (-1)^{k} \beta_{ij}^{(k)}(x') \frac{K_{mj}^{(k)}(x', x')}{P_{km}(x', x')} = a_{ij}^{(k)}(x'),
$$

where $k=1,2; j=1,2,3$, and as we mentioned above the numbers $j, l, m$ form a permutation of numbers 1,2,3. For the further regularization we replace the expression under the integral sign using boundary conditions (2) what is principally new in the regularization of singular necessary conditions. So, we get two regular relationships ($k=1,2$):

$$
\sum_{j=1}^{3} \left( \beta_{ij}^{(1)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \big|_{\xi_1=\gamma_1(\xi')} + \beta_{ij}^{(2)}(\xi') \frac{\partial u(\xi)}{\partial \xi_j} \big|_{\xi_1=\gamma_2(\xi')} \right) = \\
= \int_{S} \frac{u(\zeta)}{2\pi \cos(\nu, \zeta)} \frac{d\zeta'}{\cos(\nu, x_3)} \times \\
\times \int_{S} \sum_{m=1}^{2} \left( \frac{\alpha_{ij}^{(m)}(x') Q_{m}(\xi', x')}{{2\pi \left| x' - \xi' \right|^{2} \left| x' - \zeta' \right|^{2} P_{m}(x', \zeta')} \big|_{x' = \nu, x_3}} \right) \frac{dx'}{{2\pi \left| x' - \xi' \right|^{2}}}. \tag{11}
$$

**Theorem 3.** Let the conditions of Theorem 1 hold true. If system (10) is uniquely resolved, the conditions (2) are linear independent, the coefficients $\alpha_{ij}^{(k)}(x')$ for $i = 1,2; j = 1,3; k = 1,2$, belong to some Hölder class and the rest of the coefficients and kernels are continuous functions, functions $f_i(x')$, $i=1,2$, are continuously differentiable and vanish on the boundary $\partial S = S \setminus S$ then the relationships (11) are regular.

Let us introduce the designations:

$$
A_{ij}(x') = - \left[ \sum_{m=1}^{2} \alpha_{jm}^{(1)}(x') \frac{\partial}{\partial x_m} \right] \big|_{x' = \nu, x_3}, \ i, j = 1,2.
$$

Thus, we have established the following

**Theorem 4.** If the assumptions of Theorem 3 and conditions (12) hold true then boundary-value problem (1)-(2) is reduced to a two-dimensional system of linear integro-differential equations with Dirichlet’s condition (3) on the boundary $\partial S = S \setminus S$. Finally, there has been established

**Theorem 5.** If the assumptions of Theorem 4 hold true then boundary value problem (1), (2), (3) has Fredholm property (Fredholm property).

**Keywords:** Nonlocal conditions, three-dimensional Helmholtz equation, necessary conditions, solvability, regularization, Fredholmness.

**AMS Subject Classification:** 35J25.

**References**


NEW SWEEP ALGORITHM FOR SOLVING OPTIMAL CONTROL PROBLEM WITH MULTI-POINT BOUNDARY CONDITIONS

M.M. MUTALLIMOV\textsuperscript{1}, L.I. AMIROVA\textsuperscript{1}, F.A. ALIEV\textsuperscript{1}, SH.A. FARADJOVA\textsuperscript{1}

\textsuperscript{1}Institute of Applied Mathematics of Baku State University, Baku, Azerbaijan

e-mail: mmutallimov@bsu.edu.az

\textbf{Abstract.} A new sweep algorithm for solving the linearly-quadratic problem (LQP) of optimization with three-point unseparated boundary conditions is given, which solves a more general problem than [6]. An example is given, the solution of which gives an accurate result and corrects technical inaccuracies missed earlier by the authors [3, 4].

\textbf{Keywords:} Sweep algorithm, optimization, three-point boundary conditions, Euler-Lagrange equations.

\textbf{AMS Subject Classification:} 49J15, 49M25, 49N10.

1. Introduction

In work [6], an algorithm was developed for solving LQP optimization with multipoint unseparated boundary conditions. And in [3] a counterexample is given, in which it is shown that the results in [6] are non-optimal. However, scrupulous analysis showed that in [6] the case was considered when not all the nodal points (points in the boundary conditions) are not taken into account in the minimized quadratic functional (this is not mentioned in [6]). Note that, the statement of the problem is more general in [4], therefore, the example considered in [3, 4] cannot be solved directly using the sweep algorithm [6]. Therefore, in this paper, using the results of [4, 6], a new sweep algorithm is given for solving the LQP optimization with three-point boundary conditions. Since the multipoint problem is more cumbersome and requires special investigation, we will be content with the LQP optimization of three-point problems. It is shown that, because of the technical error in [4], the final results of [3] are also not optimal, in spite of the fact that the minimum value of the functional is less than [6]. Correcting these errors, we obtained the result for the example [3, 4], which is optimal.

2. Problem statement

In [6] the problem of LQP optimization with multipoint unseparated boundary conditions is considered. In this note, on the base of the algorithm [4], that increases the dimensionality of the original system, a new sweep method is proposed, which differs partly from the method [6]. To develop this algorithm, we used the results of [4, 6]. In order to avoid cumbersome calculations, we are satisfied with the three-point LQP optimization from [4, 6]

\[ E (i + 1) = \psi (i) E (i) + \Gamma (i) u (i) , \quad (1) \]
Step 3. The matrices constant matrices, such, that the system (2) satisfies the Kroneker-Kapelli condition [2, 4].

Using the corresponding Euler-Lagrange equations [2, 4, 5] and the known formulas from [4, 6], the following algorithm can be proposed.

Step 1. \( \psi (i), \Gamma (i), \Phi_1, \Phi_2, \Phi_3, q, R(i), C(i) \) are given and \( (i) = \Gamma (i)!^{-1} \Gamma' (i) \) are calculated.

Step 2. The matrices iterations
\[
Q (i, j) = [E + (i, j) R (i + j)]^{-1},
\]
\[
\psi (i, j) = \psi (i + j - 1) Q (i, j - 1) \psi (i, j - 1), \; \psi (i, 1) = \psi (i),
\]
\[
(i, j) = (i + j - 1) + \psi (i + j - 1) Q (j, i - 1) \psi (i + j - 1), \; (i, 1) = (i),
\]
\[
R (i, j) = R (i, j - 1) + \psi' (i, j - 1) R (i, j - 1) Q (j, j - 1) \psi (i, j), \; R (i, 1) = R (i).
\]

are calculated.

Step 3. The matrices \( H (s + 1, l - s - 1) = [E + M (s) R (s + 1, l - s - 1)]^{-1} \) are calculated.

Step 4. The following matrices
\[
T_1 (s) = R (s) + \psi' (s) R (s + 1, l - s - 1) H (s + 1, l - s - 1) \psi (s),
\]
\[
T_2 (s) = \psi' (s) H' (s + 1, l - s - 1) \psi (s + 1, l - s - 1)
\]

are calculated.

Step 5. Are calculated following matrices
\[
H_1 (s) = \Phi_2 + \Phi_3 \psi (s + 1, l - s - 1) H (s + 1, l - s - 1) \psi (s),
\]
\[
H_2 (s) = \Phi_3 \left[ \psi (s + 1, l - s - 1) H (s + 1, l - s - 1) M (s) + \psi' (s + 1, l - s - 1) M (s + 1, l - s - 1) \right] \Phi_3.
\]

Step 6. The following matrices
\[
1 (s) = \Phi_1 + H_1 (s) [E + M (0, s) T_1 (s)]^{-1} \psi (0, s),
\]
\[
2 (s) = -H_2 (s) - H_1 (s) [E + M (0, s) T_1 (s)]^{-1} M (0, s) \left( \Phi_2 + "2 (s) \Phi_3 \right),
\]
\[
3 (s) = R (0, s) + \psi' (0, s) "1 (s) [E + M (0, s) T_1 (s)]^{-1} \psi (0, s),
\]
\[
4 (s) = \Phi_1 + \psi' (0, s) \left[ E - T_1 (s) [E + M (0, s) T_1 (s)]^{-1} M (0, s) \right] \left( \Phi_2 + "2 (s) \Phi_3 \right)
\]

are calculated.

Step 7. The system of linear algebraic equations
\[
\begin{bmatrix}
1 (s) & 2 (s) \\
2 (s) & 4 (s)
\end{bmatrix}
\begin{bmatrix}
E (0) \\
\nu
\end{bmatrix}
= \begin{bmatrix}
q \\
0
\end{bmatrix}
\]

are solved and \( E (0) \) and \( \nu \) are found.

Step 8. Finally, using \( E (0) \) and \( \lambda (0) = \Phi_1 \nu \) using the formulas
\[
x(i + 1) = \psi (i) x(i) + M (i) \psi' (i)^{-1} [-\lambda (i) + R (i) x(i)],
\]
\[
\lambda (i + 1) = -\psi' (i)^{-1} [-\lambda (i) + R (i) x(i)],
\]
first we find \( E(1) \) and \( \lambda(1) \), then under this scheme are alternately found the \( E(i) \) and \( \lambda(i) \) at \( i = 2, \ldots, s-1, s+1, \ldots l-1 \).

And for \( i = s \) the formulas

\[
\begin{align*}
\lambda(s+1) &= -\psi'(s)^{-1} [\Phi_2' v - \lambda(s) + R(s)x(s)], \\
x(s+1) &= \psi(s)x(s) + M(s)\psi'(s)^{-1} [\Phi_2' v - \lambda(s) + R(s)x(s)]
\end{align*}
\]

are using.

Step 9. Then according to the formula

\[ u(i) = -\Gamma'(i) \lambda(i+1) \]

the control \( u(i) \) is determined and, thus, the solution of the original problem (1) - (3) is found.

To compare the methods proposed in this note and in [4, 6], an example is considered where \( n = 1, m = 1, l = 4, s = 2, \psi(0) = \psi(1) = 1, \psi(2) = \psi(3) = 2, \Gamma(0) = \Gamma(1) = \Gamma(2) = \Gamma(3) = 1, \Phi_1 = \Phi_2 = \Phi_3 = 1, q = 1, R(0) = R(1) = R(2) = R(3) = 1, C(0) = C(1) = C(2) = C(3) = 1. \]

Based on the above algorithm, we obtained

\[
\begin{align*}
x(0) &= \frac{7}{58}, \ x(1) = \frac{5}{58}, \ x(2) = \frac{4}{58}, \ x(3) = \frac{17}{58}, \ x(4) = \frac{43}{58}, \\
u(0) &= -\frac{1}{29}, \ u(1) = \frac{3}{29}, \ u(2) = \frac{1}{29}, \ u(3) = \frac{9}{58}.
\end{align*}
\]

Hence it is clear that the result \((J \approx 0.07)\) obtained by the proposed method differs from the results of [3,4]. As can be seen, the value of the functional (3) for this example is much less than the value of the functional for the solution in [3,4] \((J \approx 0.5)\) and in [1] \((J \approx 0.8)\), which shows that the solutions of problem (1)-(3) obtained in [3, 4, 6] are not optimal. When analyzing the method described in [4], it turned out that because of a technical error, the resulting system of equations for finding the solution does not correspond to the Euler-Lagrange equations. Therefore, in spite of the fact that the solution obtained in [4] satisfies equation (1) and the boundary condition (2), it does not give the minimum value to the functional (3). Those, the solution is not optimal.

In conclusion, we note that here we assume that there is a \( \psi^{-1}(i) \). The case when \( \psi^{-1}(i) \) does not exist, requires a separate consideration.

References

OPTIMIZATION OF THE GROOVE CUTTING CONDITION BY THE VORTEX METHODS ON THE ROTATIONAL SURFACE

UGURLU NADIROV

1Azerbaijan Technical University, Baku, Azerbaijan
e-mail: n_ugurlu@mail.ru

ABSTRACT. Present work includes understanding the effects of various milling parameters such as spindle speed, feed rate, depth of cut, the circular and longitudinal feed on the surface roughness (Ra) of finished products. The experimental plan was based on the method of measuring the influence of each factor in the surface roughness. The experiments were conducted on cast iron 45 material on CNC vertical milling machine model ГФ217C5. Their values are determined from the different values of longitudinal feed of the preform.

An analysis of the mathematical models was conducted and the factors influencing the quality of the surface were identified. The optimal parameters for surface roughness is obtained as a spindle speed of 360 rpm, a vertical feed of 0.03 mm/rev., a linear feed of 0.1 mm/rev., 0.8 mm depth of cut, cooling water at a flow rate of 10 l/min.

Keywords: Groove, feed, tool, roughness, cutting.

AMS Subject Classification: ME304.

For the analysis of technological processes and their management, it is important to identify the relationships and functional dependencies between the input and output parameters of the technological system, including the structural elements of the surfaces to be treated when they are formed. This will ensure high quality manufacture of the parts (products), optimal processing conditions and production efficiency of products [1, 3, 6].

At Azerbaijan Technical University, a vortex method of grooving for O-rings, provided on the lateral surfaces of rotation, has been developed [4].

Cylindrical and conical valves are widely used oil industry. A distinctive feature of the sealing units for these valves is that the groove of sealing elastic elements – O-ring seals - is provided on the curved surfaces of the inserts (Fig. 1). Until such constructive solutions were considered low-tech. But then according to the traditional machining methods to produce on the surface of rotating parts were cut out with the help of end mill shaped cutters with three-way interrelated movements (linear movements in the direction of two horizontal axes and rotation of the workpiece around its own axis)- feeds the workpiece. However, such method does not correspond to modern engineering standards: the tool has a complex shape, its rigidity and the possible machining parameters are low and the productivity of the method is not at desirable level.

It is known that one of the indicators of the quality of manufacturing parts is the geometric quality of the formed surfaces [2, 5]. However, machining operation should ensure high productivity and low cost of manufacturing in addition to delivering workpiece of a desired quality.

For success of the manufacturing organization finding the optimum balance between higher production rate and improved quality is the most important. Quality can be expressed in terms of different attributes such as higher dimensional accuracy, form stability, surface smoothness and fulfillment of functional requirements for specific applications etc. Thus, improving the quality
of the product ensures the application of more advanced processing methods. The effect of the parameters of cutting grooves such as cutting speed, feed rate (the circular and longitudinal feed) along with their interactions on the process is studied using method of cutting elements optimization.

The aim of the work is to of quality assurance of the treated surface.

The mathematical model of the roughness of the groove surfaces was derived. Formula (1) is a mathematical model of the kinematic component of the unevenness at the bottom of the groove.

$$R_{zk2} = \frac{0.5D - h}{\cos \frac{90(f_m - f_t)}{Z(0.5D - h)}} - (0.5D - h).$$

Formula (2) is a mathematical model of the kinematic part of the irregularities along the outer surface when cutting grooving grooves with a vortex method by two tools.

$$R_{zk} = 0.5 \left[ d + b - \sqrt{(d + b)^2 - 0.5f^2_t \left( \frac{\pi D \cdot \arcsin \frac{d}{D}}{180 \cdot d} - 1 \right)^2} \right].$$

The kinematic component of the unevenness of the inner side surface of the grooves is formed simultaneously and is identical to the formation of irregularities along the outer side surface with a single pass of the tool. Given the identity of the formation of irregularities, determine the height of irregularities on the inner side surface of the groove can be determined:

Expression (3) is a mathematical model of the kinematic component irregularities on the inner lateral surface of the groove.

$$R_{zk1} = 0.5 \left[ d - b - \sqrt{(d - b)^2 - 0.5f^2_t \left( \frac{\pi D \cdot \arcsin \frac{d}{D}}{180 \cdot d} - 1 \right)^2} \right].$$

To improve the quality of groove surfaces, it is necessary to increase the number of cutting tools and reduce the amount of longitudinal feed. However, a decrease in the amount of longitudinal feed causes a decrease in the efficiency of the process of shaping the groove, since the main time
of its cutting increases. The results of the calculation of $R_{zk}$, $R_{zk1}$ and $R_{zk2}$ at different values of the longitudinal feed ($f_t = 0.1$, 0.25, 0.5, 1, 2 mm / rot) are shown in the Table 1.

Table 1.

<table>
<thead>
<tr>
<th>Height of irregularities, $\mu$m</th>
<th>Longitudinal feed, $f_t$, mm / rev.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>Rzk</td>
<td>0.0001</td>
</tr>
<tr>
<td>Rzk1</td>
<td>0.0001</td>
</tr>
<tr>
<td>Rzk2</td>
<td>0.00004</td>
</tr>
</tbody>
</table>

An analysis of the results shows that the height of the kinematic components of the irregularities formed on the inner surface of the billet is prevailing and is of the greatest importance. Therefore, when designing such technological operations, it is sufficient to take into account only the irregularities formed on the inner surface of the groove. It should be noted that the value of $R_{zk2}$ is insignificant compared to the roughness, permissible according to the drawing for the groove surface ($R_a = 1, 6\, \mu m$, $R_z = (4 - 5)R_a$). Hence the coefficient of safety margin of the technological process is of great importance. With increasing rigidity of the elements of the technological system (tool and workpiece systems), it is possible to increase the longitudinal feed and thereby improve the efficiency of grooving.

1. Conclusions

A mathematical model of circular feed has been introduced, which is functionally connected with the longitudinal feed during the formation of the groove. Modernization element of a constructive parameter of the technological system that provides a mathematical model of circular feed is determined.

The expected maximum heights of the kinematic components of the irregularities on the outer and inner side surfaces, grooves, and also on its bottom are predicted, and the ways of their reductions are shown. It is found that the greatest height of the kinematic components of the unevenness is formed at the bottom of the groove.

References

ON A TIME EVOLUTION OF THE QUADRATIC QUANTUM SYSTEMS.II

SH.M. NAGIYEV¹, A.I. AHMADOV², SH.A. AMIROVA¹

¹Institute of Physics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan
²Department of Theoretical Physics, Baku State University, Baku, Azerbaijan,
e-mail:shakir.m.nagiyev@gmail.com, ahmadovazar@yahoo.com

ABSTRACT. By using the evolution operator method, the general analytic expressions are presented for the various wave functions and the invariant operators of the non-stationary quadratic quantum systems: (i) a free quantum particle with variable mass, (ii) a quantum particle with variable mass in an alternating uniform field, and (iii) a forced quantum harmonic oscillator with variable mass and frequency. The well-known results in the literature are derived as the special cases by the analytical expressions we have received.

Keywords: Nonstationary quadratic quantum systems, evaluation operator, invariants, wave functions.

AMS Subject Classification: 03.65.-w, 03.65 Fd.

1. INTRODUCTION

The non-stationary problems of quantum mechanics are usually solved by approximate methods. Only in rare cases it is possible to solve the problem exactly. There are a number of methods to find the exact solutions of the non-stationary problems: path integral method[5], generating function method [1, 7], the invariant operator method [3,8], the space-time transformations method (see, [6]) and the evolution operator method [4, 9]. The exact solvability of the problem means that it is possible to explicitly define such mathematical and physical quantities associated with it as the energy spectrum, wave functions, invariant operators, transition probability, scattering parameters, etc. Important examples of exactly solvable systems are non-stationary quadratic quantum systems, namely a free quantum particle with variable mass, a quantum particle with variable mass $M(t)$ in an alternating uniform field, a quantum harmonic oscillator with variable mass $M(t)$ and frequency $\omega(t)$, under a force $F(t)$. They find wide application in many areas of theoretical and mathematical physics (see the literature in [2]).

In this paper we use the evolution operator method to find exact wave functions and invariant operators for the non-stationary quadratic quantum systems, mentioned above. It seems that, the evolution operator method is one of the powerful methods. It allowed us to obtain in simple and unified fashion the most general expressions for the wave functions and invariants for the system. The results obtained generalize the results known in the literature.
2. Evolution operators

We present in this section the explicit form of the evolution operators for the systems under consideration [9,10,11]:

1) a free quantum particle with a variable mass

\[
U_F(x,t) = e^{i \frac{h S_2(t) \partial_x^2}{2}} S_2(t) = \int_{0}^{t} \frac{dt'}{2M(t')}; \tag{1}
\]

2) a quantum particle with a variable mass in an alternating uniform field

\[
U_L(x,t) = e^{i \varphi_0(x,t)} e^{-S_1(t) \partial_x} e^{ihS_2(t) \partial_x^2},
\]

where

\[
\varphi_0 = \frac{1}{h} [x \delta(t) - S_0(t)], \quad S_0(t) = \int_{0}^{t} \frac{\delta^2(t')}{2M(t')} dt', \quad S_1(t) = \int_{0}^{t} \frac{\delta(t') dt'}{M(t')}; \tag{2}
\]

3) a forced quantum harmonic oscillator with a variable mass and frequency

\[
U_H(x,t) = e^{\frac{b(t) \xi(t) \partial_x}{2}} e^{ih^{-1}[M(t)(\xi(t)+\sigma_H(t))]e^{i\alpha(t)x^2} e^{b(t) \partial_x} e^{iS(t) \partial_x^2}. \tag{3}
\]

Here the real functions \( b, \alpha \) and \( S \) are expressed in terms of function \( \eta(t) \), which satisfy the differential equation

\[
\frac{d}{dt} [M(t) \dot{\eta}(t)] + M(t) \omega^2(t) \eta(t) = 0, \tag{4}
\]

in the following way

\[
b(t) = \ln \left[ \frac{\eta(t_0)}{\eta(t)} \right], \quad \alpha(t) = \frac{M(t) \dot{\eta}(t)}{2h \eta(t)}, \quad S(t) = h \eta^2(t_0) \int_{0}^{t} \frac{dt'}{2M(t') \eta^2(t')}, \tag{5}
\]

the real function \( \xi(t) \) is a solution of the classical equation of motion

\[
\frac{d}{dt} [M(t) \dot{\xi}(t)] + M(t) \omega^2(t) \xi(t) = F(t), \tag{6}
\]

and the function \( \sigma_H(t) \) is a classical action for a harmonic oscillator under a force. The formulas (1) - (3) establish the unitary relations between quadratic quantum systems considered.

3. Wave functions

The evolution of the system can be determined if the evolution operator \( U(x,t) \) and the initial wave function \( \psi_0(x) \) are known:

\[
\psi(x,t) = U(x,t) \psi_0(x). \tag{7}
\]

It is clear that to the different choices of \( \psi_0(x) \) will correspond different wave functions \( \psi(x,t) \) at time \( t \). As an example we will take \( \psi_0(x) = Ai(ax + b) \), where \( Ai(x) \) is the Airy function. Then we shall obtain the following expressions: 1) for the free quantum particle with a variable mass

\[
\psi_F(x,t) = U_F(x,t) \psi_0(x) = e^{if(x,t)} Ai(g(x,t)), \tag{8}
\]

where \( f(x,t) = ha^2 S_0(t) \left[ ax + b - \frac{1}{2} h^2 a^4 S_2^2(t) \right], g(x,t) = ax + b - h^2 a^4 S_2^2(t) \);

2) for the quantum particle with a variable mass in an alternating uniform field

\[
\psi_L(x,t) = U_L(x,t) \psi_0(x) = e^{i[\varphi_0(x,t) + f(x_1(t),t)]} Ai(g(x_1(t),t)), \tag{9}
\]

where

\[
x_1(t) = x - S_1(t); \tag{9}
\]

3) for the forced quantum harmonic oscillator with the mass \( M(t) \) and frequency \( \omega(t) \):

\[
\psi_H(x,t) = U_H(x,t) \psi_0(x) = e^{\frac{1}{2} b(t)} e^{ih^{-1}[M(t)(\xi(t)+\sigma_H(t))]e^{i\alpha(t)x^2} e^{b(t) \partial_x} e^{iS(t) \partial_x^2} Ai(g_{\text{Ren}}(x_2(t),t)), \tag{10}
\]

\[
\text{SH.M. NAGIYEV et al. ON A TIME EVOLUTION OF THE QUADRATIC ... 295}
\]
where \( x_2(t) = e^{\lambda(t)} [x - \xi(t)] \), but the functions \( f^{\text{Ren}}(x, t) \) and \( g^{\text{Ren}}(x, t) \) are obtained from the functions \( f(x, t) \) and \( g(x, t) \) by replacing \( \hbar S_2(t) \to S(t) \).

4. IN Variant OPERATORS

By definition, an invariant operator \( I(t) \) is a such time-dependent operator that commutes with the Schrödinger operator \( \hat{S}(t) = i\hbar \partial_t - \hat{H} \), i.e. \([I, \hat{S}] = 0\), where \( \hat{H} \) is the Hamiltonian of the system. The invariant operator translates one solution of the Schrödinger equation to another solution. To construct invariants of the systems under consideration, we first construct the basis invariants \( \hat{x}_0(t) \) and \( \hat{p}_0(t) \) for them, using formulas

\[
\hat{x}_0(t) = U(t)\hat{x}U^{-1}(t) = e_1(t)\dot{x} + e_2(t)\dot{p} + e_3(t),
\]
\[
\hat{p}_0(t) = U(t)\hat{p}U^{-1}(t) = d_1(t)\dot{x} + d_2(t)\dot{p} + d_3(t).
\]

Here the coefficients \( e_i \) and \( d_i \) (\( i = 1, 2 \)) have the form:

1) for the free quantum particle with a variable mass

\[
e_1 = 1, \quad e_2 = -2S_2(t), \quad e_3 = 0, \quad d_1 = 0, \quad d_2 = 1, \quad d_3 = 0;
\]

2) for the quantum particle with the variable mass in an alternating uniform field

\[
e_1 = 1, e_2 = -2S_2(t), e_3 = 2\delta(t)S_2(t) - S_1(t), d_1 = 0, d_2 = 1, d_3 = -\delta(t);
\]

3) for the forced quantum harmonic oscillator with a variable mass and frequency

\[
e_1 = -M(t)\dot{a}_2(t), \quad e_2 = a_2(t), \quad e_3 = M(t)\Delta_2(t), \quad d_1 = -M(t)\dot{a}_1(t), \quad d_2 = a_1(t), \quad d_3 = M(t)\Delta_1(t),
\]

where \( a_1 = e^{-b(t)}, \quad a_2 = -2b^{-1}S_1(t) \), \( \Delta_1 = \Delta_2 = \Delta_i = \dot{a}_1(t)\xi(t) - a_i(t)\dot{\xi}(t), \quad i = 1, 2 \). Using now the operators \( \hat{x}_0 \) and \( \hat{p}_0 \) we can construct the most general form of the \( n \)-th order invariant \( I_n(t) \) in the form (the coefficients \( A_{0mn} \) and \( C_{n0} \) are constant):

\[
I_n(t) = \sum_{m=1}^{n} Q_{nm}(t) + C_{n0},
\]

where

\[
Q_{nm}(t) = \sum_{k=0}^{m} A_{0mk} \left( \hat{x}_0^{k}(t)\hat{p}_0^{-k}(t) + \hat{p}_0^{k}(t)\hat{x}_0^{-k}(t) \right).
\]

REFERENCES


1. Introduction

Fractional differential equations are a rather popular subject and continue to develop each passing day as well as its new and updated definitions. One of that new fractional definitions is the generalized Riemann–Liouville fractional derivative operator introduced by Hilfer in [3], it is also called fractional Hilfer derivative and defined as follows

\[
\left( D^{\alpha,\beta}_{a+} y \right)(x) = \left( \pm I_{a+}^{\alpha(1-\beta)} \frac{d}{dx} \left( I_{a+}^{(1-\beta)(1-\alpha)} y \right) \right)(x), \quad x > 0,
\]

where \( \alpha, \beta \in \mathbb{R} \), \( \alpha \in (0, 1) \), \( \beta \in \mathbb{R} \) and \( \beta \in [0, 1] \), \( I \) is the classical fractional Riemann-Liouville derivative. According to this definition, the derivative is defined by two parameters \( \alpha, \beta \), where \( \alpha \) is the order and \( \beta \) is the type of the derivative. \( \beta \) plays an important role by changing the type of the derivative, namely fractional Hilfer derivative is Riemann-Liouville fractional derivative when \( \beta = 0 \), and Caputo fractional derivative when \( \beta = 1 \) and so, \( \beta \) enables to be analyzed more accurate results numerically by interpolating 0 with 1.

Operators defined by (1) satisfy the following integration by parts formula for sufficiently good functions \( f(x) \) and \( g(x) \), [10]

\[
\int_{a}^{b} g(x) \left( D^{\alpha,\beta}_{a+} f \right)(x) \, dx = - \int_{a}^{b} f(x) \left( D^{\alpha,\beta}_{b-} g \right)(x) \, dx + I_{a+}^{(1-\alpha)(1-\beta)} f(x) I_{b-}^{\beta(1-\alpha)} g(x) \bigg|_{a}^{b}.
\]

In this study, we define Sturm-Liouville problem with modified fractional Hilfer derivative and we show the self-adjointness of the operator, orthogonality of distinct eigenfunctions and reality of eigenvalues.

2. Main results

Fractional Bessel operator is defined as

\[
\mathcal{L}_\alpha = D^{\alpha,\beta}_{0+} p(x) D^{\alpha,\beta}_{1-} + \left( q(x) - \frac{\nu^2 - 1/4}{x^2} \right),
\]

where \( D^{\alpha,\beta} \) is the modified fractional Hilfer derivative defined in [10], \( \alpha \in (0, 1) \) and \( \beta \in [0, 1] \). Consider Bessel equation involving modified fractional Hilfer derivative

\[
\mathcal{L}_\alpha y(x) = \lambda y(x)
\]
where \( p(x) \neq 0, \forall x \in (0, 1) \), \( p, q \) are real valued continuous functions in interval \((0, 1]\), \( L_\alpha \in L^2(0, 1] \). The boundary conditions for the operator \( L_\alpha \) are the following:

\[
y(0) = 0,
\]

\[
c_1 I^{(1-\alpha)(1-\beta)}_{0+} p(x) I^{(1-\alpha)}_{1-} y(1) + c_2 I^{(1-\alpha)}_{x-} y(1) = 0,
\]

where \( c_1^2 + c_2^2 \neq 0 \). The fractional boundary-value problem \((4 - 6)\) is defined as modified fractional Hilfer Sturm-Liouville problem for Bessel operator.

**Theorem 1.** Modified fractional Hilfer Bessel operator \((3)\) is self-adjoint \((0, 1]\).

**Proof.** Considering the following equation

\[
\phi(x) L_\alpha \phi(x) = \varphi(x) D^{\alpha,\beta}_{0+} p(x) D^{\alpha,\beta}_{1-} \phi(x) + q(x) \phi(x) \phi(x),
\]

\[
\phi(x) L_\alpha \varphi(x) = \phi(x) D^{\alpha,\beta}_{0+} p(x) D^{\alpha,\beta}_{1-} \varphi(x) + q(x) \phi(x) \varphi(x).
\]

Subtracting from each other of the last two equality, and integrating from 0 to 1, then we have

\[
\int_0^1 \varphi(x) L_\alpha \phi(x) \, dx - \int_0^1 \phi(x) L_\alpha \varphi(x) \, dx = \int_0^1 \left[ \varphi(x) D^{\alpha,\beta}_{0+} p(x) D^{\alpha,\beta}_{1-} \phi(x) - \phi(x) D^{\alpha,\beta}_{0+} p(x) D^{\alpha,\beta}_{1-} \varphi(x) \right] \, dx.
\]

Now, using \((2)\) and applying boundary conditions \((5) - (6)\), then we have

\[
\int_0^1 \varphi(x) L_\alpha \phi(x) \, dx - \int_0^1 \phi(x) L_\alpha \varphi(x) \, dx = 0,
\]

\[
\int_0^1 \varphi(x) L_\alpha \phi(x) \, dx = \int_0^1 \phi(x) L_\alpha \varphi(x) \, dx,
\]

\[
\langle L_\alpha \varphi, \phi \rangle = \langle \varphi, L_\alpha \phi \rangle.
\]

The proof is completed.

**Theorem 2.** The eigenvalues of modified fractional Hilfer Sturm-Liouville problem \((4 - 6)\) are real.

**Proof.** Assume that \( \lambda \) eigenvalues are complex and if we use the self-adjointness of the operator \( L_\alpha \), then we have

\[
\langle L_\alpha \varphi, \tilde{\varphi} \rangle = \langle \varphi, L_\alpha \tilde{\varphi} \rangle,
\]

\[
\langle \lambda \varphi, \tilde{\varphi} \rangle = \langle \varphi, \lambda \tilde{\varphi} \rangle
\]

\[
(\lambda - \overline{\lambda}) \langle \varphi, \tilde{\varphi} \rangle = 0.
\]

Since \( \langle \varphi, \tilde{\varphi} \rangle \neq 0 \),

\[
\lambda = \overline{\lambda}
\]

and hence \( \lambda \) eigenvalues are real.

**Theorem 3.** The eigenfunctions corresponds to different eigenvalues of modified fractional Hilfer Sturm-Liouville problem \((4 - 6)\) are orthogonal.

**Proof.** Let \( \lambda \) and \( \mu \) are two different eigenvalues corresponding to eigenfunctions \( \varphi(x) \) and \( \phi(x) \) respectively for the problem \((4 - 6)\),

\[
L_\alpha \varphi(x) = \lambda \varphi(x),
\]

\[
L_\alpha \phi(x) = \mu \phi(x).
\]
Multiply last two equations to φ(x) and ϕ(x) respectively, subtracting from each other and integrating from 0 to 1, since the self-adjointness of the operator $L_\alpha$, we have

$$(\lambda - \mu) \langle \varphi, \phi \rangle = 0.$$ 

Since $\lambda \neq \mu$,

$$\langle \varphi, \phi \rangle = 0,$$

and the proof completes.

Furthermore, we investigate the representation of the solution for the equation (4).

**Keywords:** Fractional, Bessel operator, eigenvalues, Hilfer.

**AMS Subject Classification:** 26A33, 34A08.

**References**


MATHEMATICAL MODEL FOR CALCULATING THE DYNAMIC SEAL ASSEMBLY AND THE PROBLEM OF OPTIMIZATION OF ITS PARAMETERS

A.B. PASHAYEV¹, E.N. SABZIEV²

¹Institute of Control Systems of ANAS, Baku, Azerbaijan
²Kiber Ltd Company, Baku, Azerbaijan
e-mail: adalat.pashayev@gmail.az, elkhan@kiber.az

1. INTRODUCTION

Jet apparatuses that operate on the kinetic energy of fluid or gas flow are widely used in solving various problems of oil production [1]. One such application is removing sand plugs from oil wells. At present, there are various realizations of the concept of a jet apparatus for evacuating sand plugs [2]. The design solution for a single-pipe jet apparatus can be described as follows. Water is pumped into the well under high pressure, with the jet directed at the sand plug. The pressure of the water jet loosens the plug and the resulting mixture of loosened particles and water is sent upwards to the wellhead by water driven by the ejector effect. Water is fed through the annular space separated by a dynamic seal assembly.

Development of jet apparatuses of the described design entails a number of mathematical problems, including calculation of the dynamic seal assembly and optimization of its parameters. In this paper, we present a mathematical model of the dynamic seal assembly and deal with the problem of optimal choice of its parameters.

2. DESIGN OF THE DYNAMIC SEAL ASSEMBLY AND ITS OPERATING PRINCIPLE

A packer is a reinforced rubber sleeve that serves to separate the parts of the wellbore vertically and seal off individual sections of the casing string to separate the areas of the annular space located above and below the packer. Dynamic seal assemblies are also used to separate the wellbore into sections. By a dynamic seal assembly we understand a battery of the same type sleeves.

Dynamic seal assembly is characterized by the dimensions of its sleeves, their number, and spacing. When calculating the dynamic seal assembly, the gap between the seal and the inner wall of the casing string is also taken into account.

The sealing principle is based on the loss of hydraulic head when the fluid moves through vessels of variable cross-section. Since the battery of the same type sleeves forms a certain annular slit between the pipes, which is a succession of widening and narrowing sections, a loss of its pressure occurs when the fluid passes through them. In the following paragraphs, we build a mathematical model for estimating the loss of fluid pressure on the basis of the dimensions of the gap and the specific characteristics of the fluid.

*This research was supported by grant of the Scientific Fund of SOCAR "Development of a software for computer modeling and calculation of jet equipment for cleaning sand plugs in oil wells", 2017.
3. Mathematical model of the dynamic seal assembly

The proposed system for removing sand plugs consists of a flow string, a jet apparatus attached to the end of the string and a dynamic seal assembly to separate the bottomhole zone from the annular space. As previously stated, an annular slit forms between the inner wall of the casing and the outer wall of the production tubing. To build a mathematical model of the dynamic seal assembly, the following notations were adopted, which are assumed to be preset:

1) \(D_2\) - inside diameter of the casing;
2) \(\mu\) - coefficient of dynamic viscosity of the fluid (water);
3) \(\gamma\) - specific weight of the flushing fluid.

The mathematical model of the dynamic seal assembly relates the dimensions of the sleeve to the spacing between sleeves in the battery.

Assume that the outside diameter of the production tubing \(D_1\), the outside diameter of the sleeve \(D_3\), (obviously, its inside diameter should coincide with the outside diameter of the production pipe \(D_1\)), the thickness of one sleeve \(L\), the number of sleeves forming the battery of the dynamic seal assembly \(N\).

On the basis of the input data, we determine the gap between the seal assembly and the inner wall of the casing string \(s\), the area of the gap \(F_1\) and the area of the wide part of the seal assembly between sleeves \(F_2\):

\[
s = \frac{1}{2}(D_2 - D_3),
\]

\[
F_1 = \frac{\pi}{4} s(D_2 - s),
\]

\[
F_2 = \frac{\pi}{4}(D_2 - D_1)(D_2 + D_1).
\]

According to [1], the head loss due to a sudden contraction of the passage is calculated from the formula (when the fluid passes through the gap)

\[
\Delta P^{(1)} = \frac{\gamma \zeta}{2g} \left( \frac{Q}{F_1} \right)^2,
\]

de\(\zeta = \frac{1}{2} \left( 1 - \frac{F_2}{F_1} \right)\) is the resistance coefficient, \(Q\) is the rate of flow through the battery of the dynamic seal assembly, \(g\) is the gravitational acceleration.

The head loss due to the fluid passing through the annular slit between the casing and the production tubing of length \(L\) is calculated from the formula

\[
\Delta P^{(2)} = \frac{12 \mu Q L}{\pi (D_2 - s)s^3}.
\]

Finally, in accordance with the Borda-Carnot equation, the head loss due to a sudden expansion of the pipeline can be calculated from the formula

\[
\Delta P^{(3)} = \frac{4\gamma Q^2}{g F_1^2} \zeta^2.
\]

Adding \(\Delta P^{(1)}\), \(\Delta P^{(2)}\) and \(\Delta P^{(3)}\) together and multiplying the result by the number of sleeves in the battery, we get the formula that relates the total fluid loss, the flow rate, the number of sleeves and the length of the annular slit between the casing and the production tubing:

\[
\Delta P = N \left\{ \frac{\gamma \zeta (1 + 8 \zeta)}{2g} \left( \frac{Q}{F_1} \right)^2 + \frac{12 \mu Q L}{\pi (D_2 - s)s^3} \right\}\]
4. Problem of optimizing the parameters of the dynamic seal assembly

Let us describe the requirements for the calculation of the dynamic seal assembly and formulate the problem of its optimization. The mathematical model allows us to find the optimum dimensions of the packer to use in the given downhole conditions.

Some of the parameters in the formula $\Delta P$ are determined by the design features of the well being cleaned. These include the inside diameter of the casing ($D_2$), the outside diameter of the production tubing ($D_1$), the coefficient of dynamic viscosity and specific weight of the flushing fluid ($\mu$, $\gamma$), and the required head loss determined by the operating pressure differential created by the pump and the statistical pressure in the wellbore ($\Delta P$).

Since the dynamic seal assembly is mounted along the pipe of the lower part of the jet apparatus, there is a maximum length constraint:

$$N \times L \leq L_0,$$

where $L_0$ is the set number. The dynamic seal assembly cost functional acts as the criterion of the optimal choice of parameters. The production cost of one sleeve is a nonlinear function $f(L)$ of the sleeve thickness and is set by the manufacturer.

The optimization problem can be formulated as follows. Determine the thickness of one sleeve $L$, which meets the dynamic seal assembly length constraint, satisfies the equation with respect to $\Delta P$ and minimizes the functional

$$\Im(L) = Nf(L) \rightarrow \min.$$  

Thus, the problem of optimizing the dynamic seal assembly can be formulated as follows: “Determine $L$, such that simultaneously satisfies equation (1), condition (2) and minimizes functional (3).”

The number $N$ can be expressed through $L$. Then we can obtain an explicit form for the constraint $L$:

$$L \leq \frac{\gamma_0(1+8c)}{2g} \left(\frac{Q}{F_1}\right)^2 L_0 \frac{\Delta P - \frac{12 \mu Q L_0}{\pi (D_2-s) s^3}}{\Delta P}.$$  

In this case, the explicit form of the functional $\Im(L)$ will be:

$$\Im(L) = \frac{\Delta P f(L)}{\frac{\gamma_0(1+8c)}{2g} \left(\frac{Q}{F_1}\right)^2 + \frac{12 \mu Q L}{\pi (D_2-s) s^3}}.$$  

It is obvious that the obtained expression is a smooth function of one variable, and its minimum within the indicated limits can be investigated by standard methods of differential calculus theory.

**Keywords:** Mathematical model, dynamic seal assembly, packet, head loss.

**AMS Subject Classification:** 93A30.

**References**

A FORMALIZED APPROACH TO VERIFY GPGPU APPLICATIONS.
PART 1

POGORILYY S.D.1, KRYYVI S.L.2, SLYNKO M.S. 3

1Head of Computer Engineering Department of Faculty of Radio Physics, Electronics and 
Computer Systems at Taras Shevchenko National University of Kyiv, Ukraine
2Faculty of Computer Science and Cybernetics at Taras Shevchenko National University of 
Kyiv, Ukraine
3Computer Engineering Department of Faculty of Radio Physics, Electronics and Computer 
Systems at Taras Shevchenko National University of Kyiv, Ukraine

1. INTRODUCTION

Modern stage of the software systems development is characterized by a significant compli-
cation of their development process. Existing quality control methods, on the contrary, are 
characterized by incompleteness, high complexity and insufficient reliability.
Testing process has always been the main method of increasing the reliability of programs, 
developed using traditional methods. Edsger Dijkstra once said: Program testing can be used to 
show the presence of bugs, but never to show their absence!. In addition, testing can not detect 
typical synchronization errors of parallel programs. Parallel programs may for years retain errors 
that manifest themselves after a long usage period as a reaction to a specific combination of 
numerous factors that have arisen (for example, due to the unpredictable rates of individual 
threads/processes execution in parallel programs).
However, if any of the system properties can be expressed formally, for example, in the form 
of a mathematical logic formula, then analysis of this property can be performed by verification 
methods. Normally verification of the system consists of the following parts:
(1) Construction of a mathematical model of the system under analysis.
(2) Definition of the properties to be checked in the form of a formal text (also known as 
specification).
(3) Building a formal proof of the presence or absence of the property being verified.

Usually, mathematical model of a system is a graph whose vertices are called states, and 
represent situations (or situation classes), in which the system may be present at different 
times; whose edges can have labels depicting the actions system can perform. The functioning 
of the system in this model is represented by transitions along the edges of the graph from one 
state to another. If the passable edge has a label, then this label represents the action of the 
system, executed when passing from the state at the beginning of the edge to the state at its 
end.

On the one hand, system model should not be overspecified, because the excessive model 
complexity may cause significant computational problems during its formal analysis. On the 
other hand, system model should not be oversimplified: it should reflect those aspects of the 
system that are relevant to the properties being verified, and preserve all the properties of the 
simulated system which are of interest for analysis.
Model checking (MC, [2]) approach is used to find a formal proof that the model does not meet its specification. In this paper we propose a new method of justifying that the model satisfied the specification. Our method uses the apparatus of transitive systems (TS, [1]) which allows creating abstractions of different level that can be converted into Petri nets or support temporal modeling. Verification of application for graphic video adapters (using the example of NVidia video adapters) was chosen as the subject of the study. This area perfectly illustrates the impossibility of manual verification, as the number of threads allocated to solve the problem is measures in hundreds of thousands (in Pascal/Volta architectures).

2. CUDA EXECUTION MODEL DEVELOPMENT

In this paper we decompose the computing architecture of NVidia CUDA and explore the instructions execution scheduling. Recall that at the hardware level, NVidia GPUs consist of a set of streaming multiprocessors that operate with thread sets - warps, using the SIMT execution model, which is similar to the SIMD class in Flynn taxonomy. Thus, we distinguish the following subsystems:

1. Modeling abstract instruction. In this paper we take into account two types of instructions: arithmetic and memory access. This subsystem is represented by the transition system TS1.
2. Modeling warp, which receives a set of instructions, and performs them sequentially. It is modeled by the TS2 system.
3. Modeling execution of the stream block on the current multiprocessor. The main structural unit in this process, by definition, is the warp scheduler. Subsystem is modeled by the TS3 system.

Let's consider in more detail the inner structure of each of the systems. The TS1 system consists of the following places and transitions:

- b0 - fixing the start of execution;
- p1 - preparation for the arithmetic operation;
- b1 - fixing the start of the arithmetic operation;
- p2 - operation execution;
- b2 - fixing the finish of arithmetic operation execution;
- p3 - transition to the instruction completion state;
- p4 - the formation of a memory access request;
- b3 - fixing the start of the memory access request execution;
- p5 - the process of obtaining data;
- b4 - fixation of data acquisition;
- p6 - transition to the instruction completion state;
- b5 - completion of the instruction.

The TS2 system is described by the following scheme:

- a0 - fixing the state of the warp deactivation;
- r1 - activation of the warp by the scheduler (warp gets access to multiprocessor resources for executing one instruction);
- a1 - fixing the state of the active warp;
- r2 - preparation for the instruction execution;
- a2 - fixing the state of the warp during the execution of the instruction;
r3 - receive confirmation of the execution of the instruction;
a3 - the instruction is executed;
r4 - turn off the warp.

Finally, the TS3 system simulating the execution of the stream block on the multiprocessor is described below:

c0 - fixing the planning start;
q1 - warp and instruction selection;
c1 - a warp-instruction tuple is selected;
q2 - the beginning of the instruction execution for the selected warp;
c2 - fixing the state of waiting for the instruction execution;
q3 - receiving confirmation of the execution of the instruction;
c3 - fixing the completion of the warp-instruction tuple;
q4 - go to the next iteration.

In order to describe the protocol of interaction between subsystems, the operation of the TS synchronous product is used. Only a subset of all possible global transitions is indicated, which simulates synchronized changes in the global state of the system. In our case it is expedient to determine the following global transitions of the subsystems 1, 2, 3 product:

\[ T = \{ (\varepsilon, r_1, q_1), (p_1, r_2, q_2), (p_2, \varepsilon, \varepsilon), (p_3, r_3, q_3), (p_4, r_2, q_2), (p_5, \varepsilon, \varepsilon), (p_6, r_3, q_3), (\varepsilon, r_4, \varepsilon), (\varepsilon, \varepsilon, q_4) \} \]

3. Conclusions

A formal verification approach that offers all the advantages of model checking (automatic verification, completeness), and also allows the use of tools and methods for analyzing Petri nets is proposed. The execution process of applications in NVidia CUDA architecture has been decomposed. A generalized model of this process, created in the apparatus of transitive systems, is developed as an example of the approach.

Keywords: NVidia CUDA, GPGPU, Petri net, transition system.

AMS Subject Classification: 68Q85.

References

A FORMALIZED APPROACH TO VERIFY GPGPU APPLICATIONS.
PART 2

POGORILYY S.D., KRYVYI S.L., SLYNKO M.S.

1 Head of Computer Engineering Department of Faculty of Radio Physics, Electronics and Computer Systems at Taras Shevchenko National University of Kyiv, Ukraine
2 Faculty of Computer Science and Cybernetics at Taras Shevchenko National University of Kyiv, Ukraine
3 Computer Engineering Department of Faculty of Radio Physics, Electronics and Computer Systems at Taras Shevchenko National University of Kyiv, Ukraine

1. CONSTRUCTING A PETRI NET FROM TRANSITION SYSTEM PRODUCT

Recall that in the first part of the paper decomposition of the NVidia CUDA computing architecture was made and the instructions execution scheduling mechanism was explored. The following subsystems were distinguished:

(1) Modeling abstract instruction. In this paper we take into account two types of instructions: arithmetic and memory access. This subsystem is represented by the transition system TS1.
(2) Modeling warp, which receives a set of instructions, and performs them sequentially. It is modeled by the TS2 system.
(3) Modeling execution of the stream block on the current multiprocessor. The main structural unit in this process, by definition, is the warp scheduler. Subsystem is modeled by the TS3 system.

And also the following global transitions of TS 1, 2, 3 product were determined:

\[ T = \{(\varepsilon, r_1, q_1), (p_1, r_2, q_2), (p_2, \varepsilon, \varepsilon), (p_3, r_3, q_3), (p_4, r_2, q_2), (p_5, \varepsilon, \varepsilon), (p_6, r_3, q_3), (\varepsilon, r_4, \varepsilon), (\varepsilon, \varepsilon, q_4)\} \]

Thus, a high-level system specification (a general type model) is obtained. Having the above model on hand, it becomes possible to start the process of its verification. The main models of such a process are automatic and network models. In this paper we consider network models, namely, Petri nets (PN), for which an essential apparatus of analysis methods has been developed. It is described in [1] that the semantics of the product of the TS and the semantics of the PN that models it are consistent in the sense that the sequence of global transitions \( t_1, t_2, \ldots, t_k \) is the global history of the product of the TS A if and only if it is an admissible sequence of transitions in the corresponding PN.

Accordingly, the elements of the set T become transitions of the PN, and the global states of the TS (the set of states of each of the TSs participating in the synchronous product before or after the global transition) become places of the obtained network. In our example, the following Petri net was obtained after carrying out the indicated transformations:
Once the Petri net is constructed, the problem of checking its correctness arises. In scope of this paper we consider checking the survivability of such a system. The "survivability" characteristic means that in the received model all the transitions will take part in the process of its functioning. If certain transitions in the PN never fire, it means that the system design is not built correctly, or it is redundant.

2. Model verification

Place $S_2$, which semantically represents a set of instructions, is a downside of this model. The system will work only if there are tokens in this place, making impossible to cover this PN with invariants. We overcome such an issue and investigate survivability in two different ways.

However, it is still possible to check that the desired final markup is reachable from the given initial markup, and whether all transitions are triggered in this process. The semantically correct initial markup is $M_0 = (1, 1, 0, 0, 0, 0, 0, 1)$, which means there is a single warp scheduler (token at location $S_1$), a single instruction for execution (token at place $S_2$), and there is no instruction being currently processed (token at location $S_9$). The desired final markup is $M_k = (1, 0, 0, 0, 0, 0, 0, 1)$, that is, all instructions are executed, there is no active warp, no more planning iterations occur. Let us find the solutions of the equation of state of the form:

$$A \ast x = M_k - M_0 \Rightarrow A \ast x - (M_k - M_0) = 0,$$

where $A$ is the incidence matrix of the PN. For the given PN, the state equation has the following form:

<table>
<thead>
<tr>
<th></th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
<th>$t_4$</th>
<th>$t_5$</th>
<th>$t_6$</th>
<th>$t_7$</th>
<th>$t_8$</th>
<th>$t_9$</th>
<th>$-(M_k - M_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$S_2$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$S_3$</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_4$</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$S_5$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_8$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_9$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

To solve the state equations, the TSS [2] method is applied. The following solutions were found for the matrix above:
As can be seen from the solution set, all the Petri net transitions with the given initial and final markings are alive (the value corresponding to each transition is positive at least in one of the solutions). In addition, only transitions corresponding to a single instruction type occur simultaneously.

Another verification approach is to exclude the $S_2$ place when constructing the equation of state. In this case, it is necessary to find solutions of a system of homogeneous equations of $A \ast x = 0$, based on the following matrix:

The set of solutions remains the same, regardless of the chosen approach, but using the second approach one can talk about covering the network with positive T-invariants, that is, the PN is alive under any initial markup. On the other hand, each concrete combination of initial and final markings should be considered separately if using the first approach, which reduces the generalization of the model to none.

3. Conclusion

Software testing is not a sufficient method to ensure the reliability of programs, especially when using parallel computations. On the contrary, formal verification of the mathematical model of the system under analysis provides evidence of the presence or absence of the tested properties in the system and can be carried out in a (semi) automatic mode.

In this paper we propose a formal verification approach based on a combination of transitive systems and Petri net apparatuses. The system under analysis is first presented in the form of several subsystems, allowing to determine the necessary level of abstraction. Further, the protocol of interaction between subsystems forms a set of global transitions, determining the synchronous product of subsystems. The synchronous product of the transitional systems is reduced to the Petri net, which by definition is the model of the system being analyzed. The last step allows us to use a wide range of methods for analyzing Petri nets for formal verification of the properties of the system (for example, in this paper we show a study of Petri net coverage by positive T-invariants).

Keywords: NVidia CUDA, GPGPU, Petri net, transition system.

AMS Subject Classification: 68Q85.

References

AN APPROACH TO SOLUTION TO CLASS OF COEFFICIENT-INVERSE NONLOCAL PROBLEMS FOR A PARABOLIC EQUATION

ANAR B. RAHMOV\textsuperscript{1,2}

\textsuperscript{1}The Institute of Control Systems, Azerbaijan National Academy of Sciences, 9, B. Vahabzadeh str., AZ1141, Baku, Azerbaijan
\textsuperscript{2}Baku State University, 23, Z. Khalilov str., AZ1128, Baku, Azerbaijan
e-mail: anar.rahmov@fresnel.fr, anar_r@yahoo.com

1. Introduction

The attention of researchers to nonlocal and coefficient-inverse problems has increased in the recent years [1–3]. Nonlocal form of initial and boundary conditions is caused by practical impossibility to make measurements of the state of an object (or process) in its separate points or instantly in time.

In this work, we consider the solution to classes of inverse problems in which the identifiable coefficients depend only on the time or only on the space. This specific character allows reducing the initial problems to specially built Cauchy problems with respect to a system of ordinary differential equations. It is important to note that the proposed approach does not use any iterative procedures.

2. Problems formulation

Problem 1. We consider the following inverse problem to determine an unknown coefficient $B_0 (t)$ of a linear parabolic equation:

$$\frac{\partial v (x,t)}{\partial t} = a_0 (t) \frac{\partial^2 v (x,t)}{\partial x^2} + a_1 (t) \frac{\partial v (x,t)}{\partial x} + a_2 (t) v (x,t) + f (x,t) + B_0 (t) C_0 (x,t),$$

$$(x,t) \in \Omega = \{(x,t) : 0 < x < l, \ 0 < t \leq T\},$$

under the following conditions:

$$\int_0^l e^{k \xi} v (\xi,t) \, d\xi = \psi (t), \quad k = \text{const}, \quad t \in [0, T],$$

$$v (0,t) = \psi_0 (t), \quad v (l,t) = \psi_1 (t), \quad t \in [0, T];$$

$$v (x,0) = \phi_0 (x), \quad x \in [0, l].$$

Here functions $a_0 (t) > 0, a_1 (t), a_2 (t), a_3 (t), f (x,t), C (x,t), \psi (x), \psi_0 (t), \psi_1 (t), \phi_0 (x)$ and constant $k$ are given, the functions $\phi_0 (x), \psi (t), \psi_0 (t), \psi_1 (t)$ satisfy the following consistency conditions:

$$\phi_0 (0) = \psi_0 (0), \quad \phi_0 (l) = \psi_1 (0),$$

$$\int_0^l e^{k \xi} \phi_0 (\xi) \, d\xi = \psi (0).$$
and all other necessary conditions of existence and uniqueness of the solution to the inverse problem (1)-(4). The problem (1)-(4) consists in determining the unknown continuous function \(B_0(t)\) and the corresponding solution to the boundary value problem \(v(x,t)\), which is twice continuously differentiable with respect to \(x\) and once continuously differentiable with respect to \(t\) for \((x,t) \in \Omega\), and satisfies conditions (1)-(4).

**Problem 2.** The following equation is given:

\[
\frac{\partial v(x,t)}{\partial t} = a_0(x) \frac{\partial^2 v(x,t)}{\partial x^2} + a_1(x) \frac{\partial v(x,t)}{\partial x} + a_2(x) v(x,t) + f(x,t) + B_0(x,t) C_0(x), \quad (5)
\]

\((x,t) \in \Omega = \{(x,t) : 0 \leq x \leq l, \ 0 \leq t \leq T\},

under the following conditions:

\[
k_1 v(x,0) + \int_0^T e^{k \tau} v(x, \tau) \, d\tau = \phi_0(x), \quad x \in [0, l],
\]

\[
v(0,t) = \psi_0(t), \quad v(l,t) = \psi_1(t), \quad t \in [0, T], \quad (7)
\]

\[
v(x,T) = \phi_T(x), \quad x \in [0, l]. \quad (8)
\]

It is required to determine the pair of functions \((C_0(x), v(x,t))\).

### 3. Solution to problems

In the work, we suggest a numerical approach to solving problems (1)-(4) and (5)-(8). At first, the problems with integral conditions are reduced to problems with point conditions, but with an additional unknown function as a parameter in the equation. Then, using the method of lines, the problems are reduced to a system of ordinary differential equations with unknown numerical parameters. To determine these parameters, a method is suggested based on the author’s results [4]-[8].

3.1. For example, for the problem (1)-(4), after introducing the auxiliary function

\[
u(x,t) = \int_0^T e^{k \xi} v(x, \xi) \, d\xi,
\]

and using the difference approximation of the derivatives with respect to \(t\) with the step \(h_t = T/N_t\)

\[
\frac{\partial u(x,t)}{\partial t} \bigg|_{t=t_j} = \frac{u_j(x) - u_{j-1}(x)}{h_t} + O(h_t),
\]

on the lines \(t = t_j = jh_t\), we obtain the following equations

\[
u''_j(x) + A_{1j} u'_j(x) + A_{2j} u_j(x) + f_j(x) + B_{0j} C_j(x) + A_{0j} B_{1j} = 0, \quad x \in (0, l), \quad j = 1, ..., N_t, \quad (10)
\]

with the conditions

\[
u_0(x) = \tilde{\phi}_0(x), \quad x \in (0, l), \quad (11)
\]

\[
u_j(0) = 0, \quad u_j(l) = \psi_j, \quad (12)
\]

\[
u'_j(0) = \psi_0j, \quad u'_j(l) = \tilde{\psi}_j. \quad (13)
\]

The coefficients of equations (10) are determined by means of known functions and parameters involved in the formulation of the problem (1)-(4).

For each \(j\), we seek a solution to the Cauchy problem (10)-(13) in the following form [4]-[8]:

\[
u_j(x) = \alpha_j(x) + \beta_j(x) B_{0j} + \gamma_j(x) B_{1j}. \quad (14)
\]

Here the functions \(\alpha_j(x), \beta_j(x), \gamma_j(x)\) are arbitrary for a while satisfying the conditions

\[
\alpha_j(0) = 0, \quad \beta_j(0) = 0, \quad \gamma_j(0) = 0. \quad (15)
\]
\[ \alpha_j'(0) = \psi_{0j}, \quad \beta_j'(0) = 0, \quad \gamma_j'(0) = 0. \]  
\[ (16) \]

according to (12), (13). Taking into account (14) in equation (10), owning to the arbitrariness of the functions \( \alpha_j(x) \), \( \beta_j(x) \), \( \gamma_j(x) \), we obtain three independent Cauchy problems with respect to three second-order differential equations:

\[ \begin{align*}
\alpha_j''(x) + A_{1j} \alpha_j'(x) + A_{2j} \alpha_j(x) + \tilde{f}_j(x) &= 0, \\
\beta_j''(x) + A_{1j} \beta_j'(x) + A_{2j} \beta_j(x) + \tilde{C}_j(x) &= 0, \\
\gamma_j''(x) + A_{1j} \gamma_j'(x) + A_{2j} \gamma_j(x) + A_{0j} &= 0,
\end{align*} \]

with the initial conditions (15), (16).

Solving three Cauchy problems (17), (15), (16) for each \( j \), we determine the functions \( \alpha_j(x) \), \( \beta_j(x) \), \( \gamma_j(x) \), \( j = 1, ..., N \). From the conditions (12), (13), we have an algebraic system of two linear equations:

\[ \begin{align*}
u_j(t) &= \alpha_j(t) + \beta_j(t) B_{0j} + \gamma_j(t) B_{1j} = \psi_j, \\
u_j'(t) &= \alpha_j'(t) + \beta_j'(t) B_{0j} + \gamma_j'(t) B_{1j} = \psi_{1j},
\end{align*} \]

Then, the values \( B_{0j}, B_{1j} \) are determined from the solution of these systems.

Next, by solving the Cauchy problem (10), (12), (13), we obtain the function \( v_j(x), x \in [0, l] \).

Then the procedure (46)-(51) is repeated on the line \( t = t_{j+1} \).

3.2. To solve the problem (5)-(8), the replacement

\[ u(x, t) = k_1 v(x, 0) + \int_0^t e^{k \tau} v(x, \tau) d\tau \]

and the method of lines at which the derivatives with respect to \( x \) are approximated with the step \( h_x = l/N_x \), are used. As a result, we obtain a problem with unknown parameters with respect to the system of \( N_t \) differential equations with respect to time variable.

All the necessary computational schemes, formulae, and results of the carried out numerical experiments will be given in the report.

**Keywords:** Inverse problem, parabolic equation, nonlocal conditions, method of lines, space and time dependent.

**AMS Subject Classification:** 35R30, 35K20, 65N40, 65N21.

---

Список литературы


ON STABILITY OF THE GRADIENT ALGORITHM FOR ONE SEPARABLE NONLINEAR DISCRETE OPTIMIZATION PROBLEMS*

A.B. RAMAZANOV

1Baku State University, Baku, Azerbaijan
e-mail: ram-bsu@mail.ru

ABSTRACT. We introduce the notation of steepness of a separable function of discrete argument on an ordinal-convex set. In terms of guaranteed estimates it is shown that in problems of optimization of separable coordinate-convex functions on an ordinal-convex set the gradient coordinate wise lifting algorithm is stable under small perturbations of the utility function.

Keywords: Steepness, gradient, algorithm, stability, discrete.

AMS Subject Classification: 65E05, 65E99.

The initial data many discrete optimization problems are of an approximate nature. Therefore, the analysis of stability of the solutions under instability of the parameters of the problem is of importance. Such investigations analyse the questions of stability not only of solutions of discrete optimization problems but also of algorithms to solve them (see, for example, [1], [2], [4]).

In this paper, in terms of guaranteed estimates, it is shown that in problems of maximization of separable function of discrete argument on an ordinal-convex set the gradient algorithms is stable under small perturbations of the steepness of the utility function.

Let $Z_+^n$ be a set of $n$-dimensional non-negative integer-valued (real) vectors. Let $P \subseteq Z_+^n$. In what follows we assume that the set $P$ possesses the following properties:

1. $|P| < \infty$;
2. $0 = (0,\ldots,0) \in P$;
3. $[0, x] = \{z \in Z_+^n : 0 \leq z \leq x\} \subseteq P$ for any $x = (x_1,\ldots,x_n) \in P$.

A set $P$ possessing properties 1)-3) will be referred to as a finite ordinal-convex set with zero (see, for example, [3], [4]).

For a function $F : Z_+^n \to R$ ($R$ is the set of real numbers) we the notion of the $i-$ gradient

$$\Delta_i F(x) = F(\pi_i(x)) - F(x),$$

and the $(i, j)$-gradient

$$\Delta_{ij} F(x) = \Delta_j F(\pi_i(x)) - \Delta_j F(x).$$

We consider the following discrete optimization problem $A$: find

$$\max \{F(x) = \sum_{i=1}^n F_i(x_i) : x \in P^+_f\},$$

where $F_i(x_i)$ is nondecreasing function on the set $P$.
where \( K \) is respectively. We call the gradient coordinatewise lifting algorithm stable [4], [2] for the problem \( A \).

The guaranteed (relative) estimate for the error of the gradient solving the problem \( A \), as usual, which stops at step \( \tau \) of the gradient solution of the problem \( A \), that is, the point obtained by the following gradient algorithm of coordinate wise lifting [3], [4]:

\[
x^{t+1} = \pi_i^+(x^t), \quad t = 0, 1, ..., \quad x^0 = 0 = (0, ..., 0),
\]

where

\[
\pi_i^+(x) = (x_1, ..., x_{i-1}, x_i + 1, x_{i+1}, ..., x_n),
\]

which stops at step \( \tau \) if either \( \Delta_i(x^\tau) \leq 0 \) or \( fes(x^\tau, P^-_f) = \emptyset \).

The guaranteed (relative) estimate for the error of the gradient solving the problem \( A \), as usual, a number \( \varepsilon \geq 0 \) such that

\[
\frac{F(x^*) - F(x^g)}{F(x^*) - F(0)} \leq \varepsilon.
\]

The disturbed problem \( A^\delta \) consists of the following: find

\[
\max \{ F^\delta(x) = \sum_{i=1}^n \Delta_i F^\delta(x_i) : x \in P^-_f \},
\]

where

\[
\Delta_i F^\delta(x) \leq 0, \quad 1 \leq i, j \leq n, \quad \Delta_i F^\delta(x) \leq -\alpha_i,
\]

\[
1 \leq i \leq n, \quad \alpha = (\alpha_1, ..., \alpha_n) \in R^n_+.
\]

\( F^\delta(x) \) is a nondecreasing function on the set \( P^-_f \),

\[
c(F^\delta) = c(F) + \delta, \quad \delta \in R^1_+,
\]

\[
c(F) = \min \left\{ \frac{q}{\Delta_i F(0)} : \Delta_i F(0) > 0, \quad i \in fes(0, P^-_f) \right\}, \quad \text{if } I^+(F) \neq \emptyset
\]

and \( c(F) = 1 \), if \( I^+(F) = \emptyset \),

\[
I^+ = I^+(F) = \{ i : \Delta_i F(\pi_i(0) \geq 0, \quad i \in fes(0, P) \}.
\]

The class of problems of type \( A^\delta \) is not empty [4].

Let \( \varepsilon(\delta) \) and \( \varepsilon \) be guaranteed estimates for the perturbed problem \( A^\delta \) and the initial problem \( A \) respectively. We call the gradient coordinatewise lifting algorithm stable [4], [2] for the problem \( A \) if

\[
\varepsilon(\delta) \leq K(\delta)\varepsilon,
\]

where \( K(\delta) \to 1 \) as \( \delta \to 0 \).
In essence, stability of the gradient algorithm in terms of the guaranteed estimates means separation of the class of problems for which small disturbances of the parameters of the problem (in particular, the steepness of the utility function) do not impair the guaranteed estimates for the perturbed problems.

**Theorem 2.** The gradient coordinate wise lifting algorithm is stable in the problem $A$ under small disturbances of the steepness of the utility function.

**Example.** Consider the problem to find

$$\max \{ F(x_1, x_2) = -x_1^2 + 6x_1 - x_2^2 + 5x_2 : f(x_1, x_2) = x_1^2 + x_2^2 - x_1 - x_2 \leq 0 \}.$$  \hspace{1cm} (1)

It is obvious that

$$P_f^- = \{(0,0), (0,1), (1,0)\}, \quad c(F) = \min\{2/5, 2/4\} = 2/5.$$

We construct the disturbed problem (1) by the rule given in the property in [4].

$$\max \{ F^\delta(x_1, x_2) = -x_1^2 + 6x_1 - x_2^2 + 5x_2 - (x_1 + x_2)/2 : f(x_1, x_2) = x_1^2 + x_2^2 - x_1 - x_2 \leq 0 \}.$$

It is obvious that $F^\delta(x)$ is a nondecreasing function on the set $P_f^-$, $I^+(F) \neq \emptyset$. Then

$$c(F^\delta) = \min\{2/4, 4/7\} = 1/2.$$

Since $c(F^\delta) = c(F) + \delta$, and $\delta = 0.1$, by the theorem the gradient algorithm is tolerant in problem (1) to disturbances of the steepness of the utility function $F(x)$.

**References**


CONTROL AND OPTIMIZATION OF FLUID MOTION IN MICROCRACK CHANNELS BY MEANS OF INDUSTRIAL APPLICATIONS

E.E. RAMAZANOVA¹, R.S. GURBANOV¹, M.A. MAMMADOVA¹, H.KH. MALIKOV¹, A.A. HAJIYEV²

¹“Geotechnological Problems of Oil, Gas and Chemistry” Scientific-Research Institute,
Baku, Azerbaijan
²Oil and Gas Research and Design Institute, SOCAR, Baku, Azerbaijan
e-mail: e.ramazan@gpogc.az, haciyev@socar.az

In the literature the considerable valuable material about movement of liquids in channels of various cross-section section is saved up. However, the analysis of numerous materials, namely monographs, textbooks, articles and other works has shown, that among these works there are no the researches devoted to movement of various liquids in channels of micron cross-section section. In the literature there are the numerous theoretical and experimental works confirming that with reduction of size of the size of cross-section section of channels quantitative inconsistency between settlement theoretical and experimental data is found out. This results from the fact that in experiments on walls of channels various layers which reduce the size of cross-section section of channels are formed. In these researches there is no common opinion about display of abnormal behavior of liquids. By working out of experimental installation and the control of process of experiments all collateral facts have been excluded, that in the presence of the collateral facts the effect ”microcrack-liquid” has not turned out. Considering this position in area of mechanics of a liquid, gas and plasma, authors have been studied movements of liquids in microcrack channels in specially developed installation with various liquids. Results of these researches have the big practical value in the field of human

For the first time it was revealed experimentally that a new “microcrack-fluid” effect was discovered in the cracks by displaying of the anomalous properties of viscous and anomalous fluids, and in particular water, kerosene, Newtonian and non-Newtonian oils. It is recommended to take into account the microcrack effect in the “fluid-medium” system to regulate and create new technical and technological processes in different fields of the industry.

The experiments were carried out in a specially designed unit with various fluids: water, viscous and anomalous oil.

Based on the experimental and theoretical generalizations on the motion of various fluids in the crack channels, the foundations of “mechanics of fluids and gases of microcrack channels in extremely low-permeable porous media” and the application of their results in various industries have been created. The critical value of the microcrack’s opening - $H_{critical}$, is determined for each fluid. It has been established experimentally that when viscous fluids move in a crack with an opening $H < H_{critical}$, non-Newtonian properties appear in the microcrack-fluid system, and non-Newtonian properties increase for anomalous fluids. The mentioned effects take place in the “microcrack-fluid” system only when $H < H_{critical}$, and when $H > H_{critical}$ these effects are absent.

When water and kerosene move in plane-parallel cracks, the critical values of crack opening $H_{critical}$ for the mentioned fluids at temperatures of 303 and 313K, respectively, are 25, 22 μm
and 65, 55 μm and for viscous and anomalous oil at temperatures of 303, 313, 323, 333 K, respectively 130, 115, 100, 90 and 160, 130, 115, 105 μm. For 0.3% solution of PAA at temperatures of 303, 313, 323, 333 K, respectively, 90, 72, 60, 48 μm, and for 0.15, 0.06 and 0.03% PAM solution at temperature 303 K, respectively, 60, 50 and 42 μm. The critical value of the crack opening for water in a radial two-dimensional crack at temperatures of 293 and 303 K was obtained, respectively, at 35 and 30 μm, and for anomalous oil at temperature of 303 K – 180 μm.

The resulting microcrack effect for a homogeneous fluid, purged from air or gas, appears as additional resistance, and influence of this effect in the motion of two and three-phase fluids in microcrack systems can intensify. The mathematical models are obtained for the rheophysical parameters of Newtonian and non-Newtonian fluids, depending on the opening of the microcrack. A technique for estimation of the crack opening was developed based on the experimental studies of the steady and unsteady motion of Newtonian and non-Newtonian fluids in a radial two-dimensional crack. The estimation of the crack opening gives the opportunity to judge about the success of various methods, as well as to avoid unreasonable activities.

For rheophysical parameters of Newtonian and non-Newtonian fluids depending on opening of microcracks are received mathematical models. On the basis of experimental researches of the established and unsteady movement Newton and Non-Newton fluids in plainly-radial crack the estimation technique opening of cracks is developed.

The results of the conducted studies require taking into account the crack effect during the evaluation of the parameters of the technological process system and “microcrack-fluid” technical devices, which has scientific and practical significance for various fields of industry, mechanical industry, instrumentation, chemical technology and medicine.

To eliminate the effect of crack opening in hydrodynamic processes, you can use powerful ultrasonic, hydrodynamic, acoustic and other waves which can be carried out by manufacturing special installations and technology.

Based on the experimental studies, the possibility of the control and optimization of the fluid motion in the channels was established. Here “the control” means the identification of the reasons for change in the technological indicators of the process, and “the optimization” - the development of measures to restore the technological indicators of the processes.

Petroleum industry. The oil recovery factor of the Azerbaijani fields is 0.11 – 0.60, and the average value is 40%.

Despite the operation for more than 100 years and the large-scale wellworks, the changes in the oil recovery factor within these limits show the presence of oil in microcracks, i.e. in ultra-low permeable porous media.

Existing technologies and undertaken wellworks are not sufficient to increase oil recovery.

It was established that thick liquids (water and oil), when moving in cracks with dimensions \( h \leq h_{kp} \), take form of anomalous liquids, and the anomalous liquids further strengthen the rheologic parameters.

The change in the mechanical properties of fluids within the motion in channels with dimensions \( H \leq H_{critical} \) their recovery within motion in channels with dimensions \( H > H_{critical} \) constitute the “microcrack-fluid” effect. Therefore, to activate fluids in motion and at rest in microcracks, the size of the channels in the state \( H \leq H_{critical} \) must be turned into the state \( H > H_{critical} \). This is achieved by using strong physical fields.

In this regard, we have a number of specific verifications:
1.1. Measures to increase the success rate using the hydraulic and hydro-acid fracturing;
1.2. Effects of vibrational waves to enhance the fluid motion to the bottom holes;
1.3. Ultrasound impact on the bottomhole zone.

Therefore, it is advisable to use formation stimulation techniques with strong physical fields for increase of the oil recovery efficiency.

Instead of using different agents (water, aqueous CO2 polymer solutions, air, etc.), it is necessary to use vibro-energy stimulation techniques in combination with various reagents. The
use of these techniques along with the transformation of the crack opening dimensions with \( h \leq h_{kp} \) in \( h > h_{kp} \) also contributes to a decrease in mechanical properties of liquids.

The introduction of these techniques also contributes to an increase in the oil recovery factor.

- During the use of chemical technologies in microcracks, the “microcrack-liquid” effect must be taken into account.
- During the design of lubrication systems of machines and mechanisms, it is necessary to take into account the effect of “microcrack-liquid”, i.e. the machines and mechanisms should additionally have nodes to prevent this effect.
- The present effect must be taken into account during the instrumental work, if devices have microchannels.
- In medicine, this effect can be used in blood vessels to prevent thrombosis, as well as removal of thrombus from the veins. The removal of thrombi must be performed by means of vibration effect in the vessels and capillaries.

**Keywords:** Well testing method, centrifugal pump, drainage zone model, packer effect.

**AMS Subject Classification:** 76B75, 93C95.

**References**


ON ASYMPTOTICS OF THE SOLUTION OF BOUNDARY VALUE PROBLEM FOR QUASI-LINEAR ELLIPTIC EQUATION IN CURVILINEAR TRAPEZOID

M.M. SABZALIEV $^{1,2}$, I.M. SABZALIEVA$^2$

$^1$ Baku Business University
$^2$Azerbaijan State University of Oil and Industry
e-mail: sabzalievm@mail.ru

1. INTRODUCTION

When studying numerous real phenomenon with nonuniform transitions from one physical phenomena to other ones, it is necessary to investigate singularly perturbed boundary value problems. There are a lot of works devoted to construction of asymptotics of the solution of different boundary value problems for nonlinear singularly perturbed nonlinear elliptic equations. Here we note the papers [1], [2]. But, in great majority of these problems the boundary value problems were considered either in bounded domains with sufficiently smooth boundaries, or in rectangular domains. It should be also noted that in the known papers, the input equations degenerate either into functional equations or ordinary differential equations.

2. MAIN PROBLEM

In the present paper, we consider a boundary value problem in a curvilinear trapezoid for a quasilinear elliptic equation degenerated into a nonlinear hyperbolic equation, and the degenerated problem has breaks on the bisectrix of the first quadrant.

Let $x = \varphi_1 (y)$, $x = \varphi_2 (y)$ be sufficiently smooth functions determined in $[a, b]$ and satisfying the following conditions:

1) $\varphi_1 (y) < \varphi_2 (y)$ for $y \in [a, b]$;
2) $\varphi_1 (y) < y$, $\varphi_2 (y) > y$ for $y \in [a, b]$;
3) $\varphi_1 (a) = a$, $\varphi_2 (b) = b$;
4) $\varphi_1 ' (y) < 1$, $\varphi_2 ' (y) < 1$ for $y \in [a, b]$.

The examples of such functions are: $\varphi_1 (y) = y - (y - a)^3$, $\varphi_2 (y) = (b - y)^3 + y$, $y \in [a, b]$.

We introduce the denotation:

$\Gamma_1 = \{(x, y)| x = \varphi_1 (y), a \leq y \leq b\}$, $\Gamma_2 = \{(x, y)| \varphi_1 (y) \leq x \leq \varphi_2 (y), y = b\}$,
$\Gamma_3 = \{(x, y)| x = \varphi_2 (y), a \leq y \leq b\}$, $\Gamma_4 = \{(x, y)| \varphi_1 (y) \leq x \leq \varphi_2 (y), y = a\}$.

In $\Omega = \{(x, y)| \varphi_1 (y) \leq x \leq \varphi_2 (y), a \leq y \leq b\}$, we consider the following boundary value problem

$$L_\varepsilon u \equiv -\varepsilon^p \left[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right)^p + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right)^p \right] - \varepsilon \Delta u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + F(x, y, u) = 0, \quad (1)$$

$$u|_{\Gamma} = 0, \quad (2)$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, $\varepsilon > 0$ is a small parameter, $p = 2k + 1$, $k$ is an arbitrary integer, $\Delta$ is a Laplace operator, $F(x, y, u)$ is the given smooth function. It is assumed that the function $F(x, y, u)$ satisfies the condition

$$\frac{\partial F(x, y, u)}{\partial u} \geq \gamma^2 > 0 \quad \text{for} \quad (x, y, u) \in (\Omega \setminus \{(x, y) \in \Omega| x = y\}) \times (-\infty, +\infty). \quad (3)$$
Our goal is to construct asymptotic expansion in a small parameter of generalized solution of boundary value problem (1), (2) from the space \( W^{\infty}_{p+1}(\Omega) \). In this connection we perform iterative processes.

In the first iteration process, the approximate solution of the equation (1) is sought in the form of \( W = \sum_{i=0}^{n} \varepsilon^{i} W_{i} \). The functions \( W_{1} (x, y) \) are determined from the following boundary value problems

\[
\frac{\partial W_{0}}{\partial x} + \frac{\partial W_{0}}{\partial y} + F(x, y, W_{0}) = 0, \quad W_{0}|_{\Gamma_{1}} = 0, \quad W_{0}|_{\Gamma_{4}} = 0; \quad (4)
\]

\[
\frac{\partial W_{i}}{\partial x} + \frac{\partial W_{i}}{\partial y} + \frac{\partial F(x, y, W_{0})}{\partial W_{0}} W_{i} = f_{i}, \quad W_{i}|_{\Gamma_{1}} = 0, \quad W_{i}|_{\Gamma_{4}} = 0; \quad i = 1, 2, ..., n. \quad (5)
\]

Here, the functions \( f_{i} \) depend on \( W_{0}, W_{1}, ..., W_{i-1} \); \( i = 1, 2, ..., n \) and their derivatives. We have the following

**Lemma 1.** Let \( F(x, y, u) \in C^{2n+2}(\Omega \times (-\infty, +\infty)) \) satisfy the conditions

\[
F(x, y, u)\big|_{x=y} = 0, \quad \frac{\partial F(x, y, 0)}{\partial x^{i} \partial y^{j} \partial u^{k}} \big|_{x=y} = 0, \quad y \in [a, b], \quad i = i_{1} + i_{2} + i_{3}, \quad i_{1}, i_{2}, i_{3} = 0, 1, ..., m. \quad (6)
\]

Then problem (4) has a unique solution, \( W_{0} (x, y) \in C^{2n+2}(\Omega) \), and

\[
\frac{\partial^{2} W_{0}(x, y)}{\partial x^{i} \partial y^{j}} \big|_{x=y} = 0, \quad y \in [a, b], \quad i = i_{1} + i_{2}, \quad i_{1}, i_{2} = 0, 1, ..., m. \quad (7)
\]

The function \( W_{0} \) satisfies the boundary conditions:

\[ W_{0}|_{\Gamma_{1}} = 0, \quad W_{0}|_{\Gamma_{4}} = 0. \quad (8) \]

The following lemma is valid.

**Lemma 2.** For every \( y_{1} \in [a, b] \) the problem (7), (8) has a unique solution that is infinitely differentiable with respect to \( \tau \), and has continuous derivatives to \( (2n+2) \)-th order inclusively, with respect to \( y_{1} \), and the function \( V_{0}(\tau, y_{1}) \) and its derivatives exponentially tend to zero as \( \tau \rightarrow +\infty \).

The problems (9), (10) for \( j = 1, 2, ..., n+1 \) from which the functions \( V_{1}, V_{2}, ..., V_{n+1} \) are successively determined, are linear and their solutions may be written in the explicit form.
Assume that the function $F(x, y, u)$ satisfies the condition
\[
\frac{\partial^k F}{\partial x^{k_1} \partial y^{k_2} \partial u^{k_3}} (\varphi_1(b), b, 0) = 0; \quad k = k_1 + k_2 + k_3; \quad k = 0, 1, ..., 2n + 1. \tag{11}
\]
Then the constructed sum $W + V$ will satisfy the boundary conditions: $(W + V)|_{\Gamma_1} = 0$, $(W + V)|_{\Gamma_4} = 0$, $(W + V)|_{\Gamma_2} = 0$. For the sum $W + V$ the boundary condition (2) need not be fulfilled on $\Gamma_3$. Therefore it is necessary to construct the boundary layer type function $\eta = \sum_{j=0}^{n+1} \varepsilon^j \eta_j$ near the boundary $\Gamma_3$, that would provide satisfaction of the boundary condition $(W + V + \eta)|_{\Gamma_3} = 0$. The construction of functions $\eta_j$ is a little different from the procedure of finding the functions $V_j; j = 0, 1, ..., n + 1$.

Assume that the function $F(x, y, u)$ satisfies the condition
\[
\frac{\partial^k F}{\partial x^{k_1} \partial y^{k_2} \partial u^{k_3}} (\varphi_2(a), a, 0) = 0; \quad k = k_1 + k_2 + k_3; \quad k = 0, 1, ..., 2n + 1. \tag{12}
\]
Then the sum $W + V + \eta$ satisfies the boundary condition (2): $(W + V + \eta)|_{\Gamma_1} = 0$.

Combining the obtained results, we arrive at the following statement.

**Theorem.** Assume that $F(x, y, u) \in C^{2n+2}(\Omega \times (-\infty, +\infty))$ and conditions (3), (6), (11), (12) are fulfilled. Then from the generalized solution of problem (1), (2) we have the asymptotic representation
\[
u = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{j=0}^{n+1} \varepsilon^j V_j + \sum_{j=0}^{n+1} \varepsilon^j \eta_j + z,
\]
where the functions $W_i$ are determined by the first iterative process, $V_j, \eta_j$ are the boundary layer functions near the boundaries $\Gamma_2, \Gamma_3$ determined by the relevant iterative processes, $z$ is a remainder term, and the following estimation is valid for it
\[
\varepsilon^p \int \int_{\Omega} \left[ (\frac{\partial z}{\partial x})^{p+1} + (\frac{\partial z}{\partial y})^{p+1} \right] dxdy + \varepsilon \int \int_{\Omega} \left[ (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 \right] dxdy + 
\]
\[
+ C_1 \int \int_{\Omega} z^2 dxdy \leq C_2 \varepsilon^{2n+2},
\]
where $C_1 > 0$, $C_2 > 0$ are constants independent of $\varepsilon$.

**Keywords:** Asymptotics, boundary layer function, remainder term.

**AMS Subject Classification:** 35J62, 35J25, 65N99.

**References**


A GRADIENT IN OPTIMAL CONTROL PROBLEM OF PROCESSES DESCRIBED BY FIRST ORDER SYSTEM

SADYGOV M. A.\textsuperscript{1}, MIRZAYEVA H.G.\textsuperscript{2}, AKHUNDOV H.S.\textsuperscript{3}

\textsuperscript{1}Baku State University, Baku, Azerbaijan
\textsuperscript{2}Baku Engineering University, Baku, Azerbaijan
\textsuperscript{3}Institut of Applied Mathematics, BDU, Baku, Azerbaijan

e-mail: misreddin08@rambler.ru, hijanamis@gmail.com

In the paper a gradient is calculated in an optimal control problem of processes described by first order system.

Let that \( a > 0, \ b > 0, \ c > 0, \ D = [0, a] \times [0, b] \times [0, c], \ D_1 = [0, b] \times [0, c], \ D_2 = [0, a] \times [0, c], \ D_3 = [0, a] \times [0, b]. \) Denote by \( f_i : D \times R^{n_1+n_2+n_3} \times R^r \rightarrow R^{n_i}, i = 1, 2, 3; \) \( \varphi_1^0 : [0, b] \times [0, c] \times R^{n_2 \times R^r} \rightarrow R, \) \( \varphi_2^0 : [0, a] \times [0, b] \times R^{n_1 \times R^r} \rightarrow R, \) \( \varphi_3^0 : [0, a] \times [0, b] \times R^{n_3 \times R^r} \rightarrow R, \) \( f_0 : D \times R^{n_1+n_2+n_3} \times R^r \rightarrow R, \) \( \varphi_1 : [0, b] \times [0, c] \times R^r \rightarrow R^{n_1}, \) \( \varphi_2 : [0, a] \times [0, c] \times R^r \rightarrow R^{n_2}, \) \( \varphi_3 : [0, a] \times [0, b] \times R^r \rightarrow R^{n_3}. \)

Assume that following conditions are fulfilled:

1) functions \( f_i, f_i(u,v,w), f_{ih_0}, i = 0, 1, 2, 3, \) are continuous on the variables in the set \( D \times R^{n_1+n_2+n_3} \times R^r \) and satisfy Lipschitz condition on variables \((u, v, w, h_0).\)

2) functions \( \varphi_1^0, \varphi_2^0, \varphi_3^0, \varphi_1, \varphi_2, \varphi_3, \varphi_1^0, \varphi_1^0_1, \varphi_2^0, \varphi_2^0_1, \varphi_3^0_1, \varphi_1^1, \varphi_2^0, \varphi_3^0_1, \varphi_3^0_1 \) are continuous on the variables in the set \( D \times R^{n_1+n_2+n_3} \times R^r \) respectively and satisfy Lipschitz condition on variables \((u, h_1), (v, h_2), \) and \((w, h_3).\)

Denote by \( V_x = \{ u \in L_x^n(D) : u_x \in L_x^n(D) \}, \) \( V_y = \{ v \in L_y^n(D) : v_y \in L_y^n(D) \}, \)

\[ V_z = \{ w \in L_z^n(D) : w_z \in L_z^n(D) \}. \]

Let \( H = L_x^n(D) \times L_y^n(D_x) \times L_y^n(D_y) \times L_z^n(D_x) \times L_z^n(D_y), h_0(\cdot) \in L_x^n(D), h_1(\cdot) \in L_y^n(D), h_2(\cdot) \in L_z^n(D), \)

A function \((u, v, w) \in V = V_x \times V_y \times V_z\) satisfying system

\[ u_x(x,y,z) = f_1(x,y,z), u_x(x,y,z), u(x,y,z), w(x,y,z), h_0(x,y,z), \]

\[ v_y(x,y,z) = f_2(x,y,z), u_x(x,y,z), u(x,y,z), w(x,y,z), h_0(x,y,z), \]

\[ w_z(x,y,z) = f_3(x,y,z), u_x(x,y,z), u(x,y,z), w(x,y,z), h_0(x,y,z) \]

and boundary conditions

\[ u(0,y,z) = \varphi_1(y,z,h_1(y,z)), v(x,0,z) = \varphi_2(x,z,h_2(x,z)), \]

\[ w(x,y,0) = \varphi_3(x,y,h_3(x,y)) \]

is called a solution of problem (1), (2).

In Lemma 1 a continuous dependence of solutions (1) and (2) on the right-hand side and the boundary condition are obtained (see also [1]).

**Lemma 1.** Let conditions 1) and 2) are satisfied. If \((\bar{u}, \bar{v}, \bar{w}) \in V_x \times V_y \times V_z\) and \(h_0(\cdot) \in L_x^n([0,1]^3), h_1(\cdot) \in L_y^n([0,1]^2), h_2(\cdot) \in L_y^n([0,1]^2), h_3(\cdot) \in L_z^n([0,1]^2)\) and inequalities are satisfied

\[ d(\bar{u}_x(x,y,z), f_1(x,y,z,\bar{u}(x,y,z), \bar{v}(x,y,z), \bar{w}(x,y,z), h_0(x,y,z))) \leq \rho_1(x,y,z), \]

\[ d(\bar{v}_y(x,y,z), f_2(x,y,z,\bar{u}(x,y,z), \bar{v}(x,y,z), \bar{w}(x,y,z), h_0(x,y,z))) \leq \rho_2(x,y,z), \]

\[ d(\bar{w}_z(x,y,z), f_3(x,y,z,\bar{u}(x,y,z), \bar{v}(x,y,z), \bar{w}(x,y,z), h_0(x,y,z))) \leq \rho_3(x,y,z), \]

\[ d(\bar{u}_x(x,y,z), u_x(x,y,z)) \leq \epsilon_1, \]

\[ d(\bar{v}_y(x,y,z), v_y(x,y,z)) \leq \epsilon_2, \]

\[ d(\bar{w}_z(x,y,z), w_z(x,y,z)) \leq \epsilon_3, \]

then the solutions to the system (1), (2) are unique and they depend continuously on those of the system (1), (2) on the right-hand side and the boundary condition.
Consider a minimization of functional
\[ J(h_0, h_1, h_2, h_3) = \int_0^b \int_0^c \varphi_0(y, z, u(a, y, z), h_1(y, z)) \, dy \, dz + \int_0^a \int_0^b \varphi_3(x, y, w(x, y, c), h_3(x, y)) \, dx \, dy + \int_0^a \int_0^b \psi_0(x, y, z, u(x, y, z)) \, dx \, dy + \int_0^a \int_0^b \psi_3(x, y, w(x, y, z)) \, dx \, dy \]
where \( \varphi_i(x, y, z) \) and \( \psi_i(x, y, z) \) are given functions, and \( a, b, c \) are fixed constants.
on the solution sets of problem (1), (2).

Denote \( h = (h_0, h_1, h_2, h_3) \), \( \psi = (\psi_1, \psi_2, \psi_3) \) and introduce the Hamilton-Pontryagin function

\[
H(x, y, z, u, v, w, h_0, \psi) = f_0(x, y, z, u, v, w, h_0) + \sum_{i=1}^{n_1} f_1^i(x, y, z, u, v, w, h_0) \psi_1^i + \sum_{i=1}^{n_2} f_2^i(x, y, z, u, v, w, h_0) \psi_2^i + \sum_{i=1}^{n_3} f_3^i(x, y, z, u, v, w, h_0) \psi_3^i,
\]

and also write out a conjugate problem of function \( \psi = (\psi_1, \psi_2, \psi_3) \):

\[
\psi_{1x} = -H_u, \quad \psi_1(a, y, z) = \varphi^0_{1u}(y, z, u(a, y, z), h_1(y, z)) \quad 0 \leq y \leq b, 0 \leq z \leq c,
\]

\[
\psi_{2y} = -H_v, \quad \psi_2(x, b, z) = \varphi^0_{2v}(x, z, v(x, b, z), h_2(x, z)) \quad 0 \leq x \leq a, 0 \leq z \leq c,
\]

\[
\psi_{3z} = -H_w, \quad \psi_3(x, y, c) = \varphi^0_{3w}(x, y, w(x, y, c), h_3(x, y)) \quad 0 \leq x \leq a, 0 \leq y \leq b.
\]

In Theorem 1 a gradient is calculated in an optimal control problem of processes described by a first order system. Analogical problem is studied in [2] for two variable problem.

**Theorem 1.** Let conditions (1) and (2) are satisfied. Then the functional (3) under conditions (1), (2) is continuous and differentiable on \( h = (h_0, h_1, h_2, h_3) \) in norm \( H \) everywhere on \( H \) and its gradient \( J'(h) \in H = L_2^s(D) \times L_2^s(D_1) \times L_2^s(D_2) \times L_2^s(D_3) \) at the point \( h = (h_0, h_1, h_2, h_3) \) is representable in the form:

\[
J'(h) = \{H_h(x, y, z, u(x, y, z, h), v(x, y, z, h), w(x, y, z, h), h_0(x, y, z), \psi(x, y, z, h)) ;
\]

\[
\varphi^0_{1h_1}(y, z, u(a, y, z, h), h_1(y, z)) + \sum_{i=1}^{n_1} \psi_1^i(0, y, z, h) \varphi^1_{1h_1}(y, z, h_1(y, z)) ;
\]

\[
\varphi^0_{1h_2}(x, z, v(x, b, z, h), h_2(x, z)) + \sum_{i=1}^{n_2} \psi_2^i(x, 0, z, h) \varphi^1_{2h_2}(x, z, h_2(x, z)) ;
\]

\[
\varphi^0_{1h_3}(x, y, w(x, y, c, h), h_3(x, y)) + \sum_{i=1}^{n_3} \psi_3^i(x, y, 0, h) \varphi^1_{3h_3}(x, y, h_3(x, y)) \}.
\]

where \((u(x, y, z, h), v(x, y, z, h), w(x, y, z, h))\) a solution of problem (1), (2), corresponding to control \( h = (h_0, h_1, h_2, h_3) \) and \( \psi = (\psi_1, \psi_2, \psi_3) \) a solution of conjugate system (4).

Note that using formula (5) the necessary optimality conditions in problem (1)-(3) for convex set \( U \subseteq H \) can be obtained.

**Keywords:** Lipschitz condition, gradient, optimality conditions.

**AMS Subject Classification:** 05C35, 52A20.

**References**


SUBDIFFERENTIAL OF THE SECOND ORDER AND CONDITION OF THE OPTIMALITY

MISRADDIN A. SADYGOV

1Baku State University, Z. Khalilov str. 23, Baku, Azerbaijan
e-mail: misreddin08@rambler.ru

1. Subdifferential of the second order

Subdifferential of the second order is studied by different authors, but at the present time there is no unequivocally a captured definition of the subdifferential. It is known that determination of differential of the second order is the analog of differential of the first order. But in definitions of the subdifferential of the second order such analog general case are not available. If we consider F. Clarke subdifferential, then we can obtain different analogues of the second order subdifferential.

Let $f : X \to R$, where $X$ - Banach space. The consider the F.Clarke (see [1]) derivative

$$f^0(x; x) = \lim_{z \to x, \lambda \to 0} \frac{1}{\lambda} (f(z + \lambda x) - f(z)).$$

By putting

$$f^{00}(x_0; x_1, x_2) = \lim_{z \to x_0, \lambda_1, \lambda_2 \to 0} \frac{1}{\lambda_1 \lambda_2} (f^0(z + \lambda_1 x_1 + \lambda_2 x_2) - f(z + \lambda_1 x_1) - f(z + \lambda_2 x_2) + f(z))$$

we can consider the second order subdifferential of the following form:

$$\partial^2 f(x_0) = \{ b \in B(X^2; R) : f^{00}(x_0; x_1, x_2) \geq b(x_1, x_2) \text{ at } (x_1, x_2) \in X^2 \},$$

where the set of all continuous symmetric bilinear functions from $X^2$ in $R$ is denoted by $B(X^2; R)$. Such a definition was first introduced by the author in 1980 a number of their properties there studied, are discussed by V.F.Demyanov and A. M. Rubinov and published only in 1988 more generally, i.e. is considered subdifferential of the arbitrary order where the derivative of the second order by the direction $f^{00}(x_0; x_1, x_2)$ replaced by

$$f^{[2]}(x_0; x_1, x_2) = \lim_{z \to x_0, \lambda_1, \lambda_2 \to 0} \frac{1}{\lambda_1 \lambda_2} (f(z + \lambda_1 x_1 + \lambda_2 x_2) - f(z + \lambda_1 x_1) - f(z + \lambda_2 x_2) + f(z))$$

and considered subdifferential of the second order (see [5],[6])

$$\partial f(x_0) = \{ b \in B(X^2; R) : f^{[2]}(x_0; x_1, x_2) \geq b(x_1, x_2) \text{ at } (x_1, x_2) \in X^2 \}.$$ 

Note that $f^{00}(x_0; x_1, x_2) \leq f^{[2]}(x_0; x_1, x_2)$. 

Although the derivatives with respect to direction $f^{00}(x_0; x_1, x_2)$ and $f^{[2]}(x_0; x_1, x_2)$ are similar, but they do not coincide in the general case. The derivative with respect to direction $f^{[2]}(x_0; x_1, x_2)$ has some good properties. Under definite conditions the function $(x_1, x_2) \to f^{[2]}(x_0; x_1, x_2)$ is bisublinear (see [5]). Close to definition of subdifferential of the second order is also considered in [2] and [3] (see also [4]).
Let’s denote $B(x, \delta) = \left\{ z \in X : \|z - x\| \leq \delta \right\}$, where $\delta > 0$. If function $f$ and derivative Frechet $f'$ satisfy Lipschitz’s condition on the set $B(x_0, \delta)$, then

$$f^{00}(x_0; x_1, x_2) = \lim_{z \to x_0, \lambda \downarrow 0} \frac{1}{\lambda} (f'(z + \lambda x_2)x_1 - f'(z)x_1).$$

In the works [5, 6] also considered another definition of the subdifferential of arbitrary order. In particular from this it follows, that

$$f^{(2)}(x_0; x_1, x_2) = \sup_{z_1, z_2 \in X} \lim_{\lambda_1, \lambda_2 \downarrow 0} \frac{1}{\lambda_1 \lambda_2} (f(x_0 + \lambda_1 z_1 + \lambda_2 z_2 + \lambda_1 x_1 + \lambda_2 x_2) - f(x_0) - \lambda_1 f(x_0 + \lambda_1 z_1) - \lambda_2 f(x_0 + \lambda_2 z_2) + f(x_0 + \lambda_1 z_1 + \lambda_2 z_2)),$$

and $\partial^{(2)} f(x_0) = \{ b \in \bar{B}(X^2; R) : f^{(2)}(x_0; x_1, x_2) \geq b(x_1, x_2) \text{ at } (x_1, x_2) \in X^2 \}$. In the paper a number of properties of the subdifferential $\partial^{(2)} f(x_0)$. In the paper also following definition of the subdifferential is considered and the number of their properties is studied. Let $x \in X$ and

$$f^{(2)+}(x_0; x) = \sup_{z \in X} \lim_{\lambda \downarrow 0} \frac{1}{\lambda^2} (f(x_0 + \lambda z + 2 \lambda x) - 2 f(x_0 + \lambda z + \lambda x) + f(x_0 + \lambda z)),$$

$$f^{(2)-}(x_0; x) = \inf_{z \in X} \lim_{\lambda \downarrow 0} \frac{1}{\lambda^2} (f(x_0 + \lambda z + 2 \lambda x) - 2 f(x_0 + \lambda z + \lambda x) + f(x_0 + \lambda z)).$$

A set $D_2 f(x_0) = \{ Q \in B_0(X) : f^{(2)+}(x_0; x) \leq Q(x) \leq f^{(2)-}(x_0; x) \text{ at } x \in X \}$ we will call bidifferential of function $f$ in the point $x_0$, where $B_0(X)$ the set of all continuous quadratic functionals.

In the paper the optimality conditions of the second order are obtained. In a nonsmooth analysis the distance function plays an important role. The author in works [5, 6] showed that at the research of the subdifferential and condition of the optimality of high order also an important role is played the degree of the distance function.

2. ABOUT NECESSARY CONDITIONS OF THE EXTREMUM OF THE SECOND ORDER

Let $X$ and $Y$ be Banach spaces, $C \subset X$, $F : X \to Y$, $S : X \to Y$, $f : X \to R$, $\varphi : X \to R$, $\alpha > 0$, $\nu > 0$, $\beta \geq \alpha \nu$, $K > 0$, $\delta > 0$, $\sigma$.

Let’s put $B = B(0, 1)$ and $d_C(x) = \inf \{ \|z - x\| : z \in C \}$, where $C \subset X$.

Let $o : R_+ \to R_+$, where $\lim_{i \downarrow 0} o(i) = 0$. The mapping $F$ is said to be $S - (\alpha, \beta, \nu, \delta, o(\beta))$ locally Lipschitz with the constant $K$ at the point $x \in E$, if $F$ satisfies the condition

$$\|F(x + z) - F(x) - S(x + z) - S(x)\| \leq K \|z\|^\sigma (\|x\|^\beta - \alpha \nu + \|z\|\delta - \alpha \sigma) + o(\|x\|^\beta)$$

at $x$, $z \in \delta B$, where $S(0) = 0$. If $S(x) \equiv 0$, then the mapping $F$ is said to be $(\alpha, \beta, \nu, \delta, o(\beta))$ locally Lipschitz with the constant $K$ at the point $x$.

Let’s put $S_\lambda(x) = f(x) - \varphi(x - x_0) + \lambda(d_D^2(x) + \|x - x_0\| d_D(x)) + o(\|x - x_0\|^2)$.

**Theorem 1.** If $x_0$ is the minimum of function $f$ on the set $C$, $f$ satisfies $\varphi(1, 2, 1, \delta, o(2))$ locally Lipschitz condition with the constant $K$ at the point $x_0$, $C \subset B(x_0, \delta)$ and $D = \{ x \in C : \varphi(x - x_0) \leq 0 \}$, then $S_\lambda^{(2)+}(x_0; x) = \lim_{i \downarrow 0} \frac{1}{\delta} (S_\lambda(x_0 + 2tx) - 2S_\lambda(x_0 + tx) + S_\lambda(x_0)) \geq 0$ at $x \in X$ and $\lambda \geq K$.

If $\varphi : X \to R$, where $\varphi(0) = 0$, and $C \subset X$, then we will put

$$T_{\alpha, \nu}(x_0; C, \varphi) = \{ x \in X : \exists o(x, \lambda) : [0, \lambda_x] \to R_+, \text{ where } \frac{o(x, \lambda)}{\lambda^\nu} \to 0 \text{ at } \lambda \downarrow 0 \text{ and } \exists \lambda_i \downarrow 0, \exists u_i \subset X, \text{ where } \frac{1}{\lambda_i^\nu} \|v_i - x\| \to 0 \text{ at } i \to +\infty, \text{ that } x_0 + \lambda_i u_i \in C, \varphi(\lambda_i u_i) \leq o(x, \lambda_i) \text{ at all } i \}.$$  

**Theorem 2.** If $x_0$ is the minimum point of function $f$ on the set $C$, there exist $\alpha > 0$, $0 < \nu \leq 2$, $\tau > 0$, $\sigma > 0$, where $\tau \geq 2 - \alpha \nu$, $\sigma \geq \frac{\delta - \alpha \sigma}{\alpha}$, finite positively homogeneous function $\varphi_1$ of degree $\tau$, function $o : R_+ \to R_+$, where $\lim_{i \downarrow 0} o(i) = 0$, numbers $\delta > 0$ and $K$.
such that \(|f(x_0 + x + z) - f(x_0 + x) - \varphi(x + z) + \varphi(x)| \leq K \|z\|^\nu (\varphi_1(x) + \|z\|^\sigma) + o(\|x\|^2)\) for \(x \in T_{\alpha,2}(x_0; C, \varphi), \|x\| \leq \delta, \ z \in X, \|z\| \leq \|x\|, \ x_0 + x + z \in C, \) then
\[
\lim_{\lambda \to 0} \frac{1}{\lambda^2} \left( f(x_0 + \lambda x) - \varphi(\lambda x) - f(x_0) \right) \geq 0 \text{ at } x \in T_{\alpha,2}(x_0; C, \varphi).
\]

Function \(f\) we will call twolipschitz with the constant \(K\) at the point \(x_0\), if \(f\) satisfies to the condition \(|f(x_0 + x) - 2f(x_0) + f(x_0 - x)| \leq K \|x\|^2\) at \(x \in \delta B, \delta > 0\). Function \(f\) we will call \((\theta, \delta)\)-strongly bilipschitz with the constant \(K\) at the point \(x_0\), if \(f\) satisfies to the condition \(|f(x_0 + x) - f(x_0 + z)| \leq K \|x - z\|^{2-\theta}\) at \(x, \delta \in \delta B, \delta > 0, 0 < \theta \leq 2\).

Let’s put \(g_k(x) = f(x) + k d_2(x)\), where \(d_2(x) = d_2^k(x)\). If from \(\lim_{t \to 0} \frac{1}{t}(d(x_0 + tx) - d(x_0)) = 0\) it follows that \(\lim_{t \to 0} \frac{1}{t}(d(x_0 + tx) - d(x_0)) = 0\), then set \(C\) we will call regular the second order at the point \(x_0\).

**Theorem 3.** If \(x_0\) is the minimum point of function \(f\) on the set \(C, f\) Lipschitz function at the neighborhood of the point \(x_0\) and twolipschitz function with the constant \(L\) at the point \(x_0\), then for any \(x \in X\) there is the number \(k > 0\) such that \(g_k(x_0; x) = \lim_{t \to 0} \frac{1}{t^2}(g_k(x_0 + tx) - 2g_k(x_0) + g_k(x_0 - tx)) \geq 0\) at \(x \in X\).

If for some \(\delta > 0\) inequality is satisfied \(|f(x + x_1 + x_2) - f(x + x_1) - f(x + x_2) + f(x)| \leq L \|x_1\| \|x_2\|\) at \(x \in x_0 + \delta B, x_1, x_2 \in X, \|x_1\| \leq \delta, \|x_2\| \leq \delta\), then function \(f\) is called 2-Lipschitz with a constant \(L\) in the neighborhood of the point \(x_0\).

Let’s put \(Q_C(x_0) = \{(x_1, x_2) \in X^2 : \|x_2\| \leq \|x_1\|, \|x_2\| \leq \|x_1\|\}\) and \(\Omega_C(x_0) = \{b \in \bar{B}(X; R) : b(x_1, x_2) \leq 0 \text{ at } (x_1, x_2) \in Q_C(x_0)\}\).

**Theorem 4.** If \(X\) is Hilbert space, \(x_0\) is the minimum point of function \(f\) on the set \(C, f\) lipshitz and 2-lipshitz function with the constant \(L\) at the neighborhood of the point \(x_0, C\) the closed convex set and regularly at the second order at the point \(x_0\) or \((\theta, \delta)\)-strongly bilipschitz function with the constant \(L\) at the point \(x_0\), then \(\sup\{b(x, x) : b \in \partial_2 f(x_0) + \Omega_C(x_0)\} \geq 0\) at \(x \in X\).

Let’s put \(g_k(x) = f(x) - (f(x_0), x - x_0) + kd_2(x)\), where \(D = \{x \in C : \langle f'(x_0), x - x_0 \rangle \leq 0\}\), \(d_2(x) = d_2^k(x)\).

**Theorem 5.** If \(x_0\) is the minimum point of function \(f\) on the set \(C, f\) Lipschitz function with the constant \(L\) at the neighborhood of the point \(x_0\), it is differentiable at the point \(x_0\) and \((f'(x_0), x) - (1, 2, \nu, \delta, o(2))\) locally Lipschitz condition with the constant \(K\) at the point \(x_0\), then for any \(x \in X\) there is number \(k > 0\) such that \(g_k^2(x_0; x) \geq 0\) at \(x \in X\).

**Theorem 6.** If \(X\) is Hilbert space, \(x_0\) is the minimum point of function \(f\) on the set \(C, f\) the condition of the theorem 5 is satisfied, \(f\) 2-lipschitz function with the constant \(L\) at the neighborhood of the point \(x_0, C\) is closed convex set, then \(\sup\{b(x, x) : b \in \partial_2 f(x_0) + \Omega_D(x_0)\} \geq 0\) at \(x \in X\).

**Keywords:** Subdifferential of the second order, bisublinear function, local minimum.

**AMS Subject Classification:** 05C35, 52A20.

**References**


EXPERIMENT OF SOLAR ENERGY PLANTS APPLICATION IN THERMAL EFFECT PROCESS OF THE OIL RESERVOIR

TULPARKHAN SALAVATOVI, FUAD MAMMADOVI

1Azerbaijan Oil and Industry University, Baku, Azerbaijan
e-mail: fm.solarpower@gmail.com

ABSTRACT. In the paper the oil wells’ thermal engineering process of by using high-powered solar energy plants was argued. The solar energy facilities with parabolic and parabola through concentrators were tested in Absheron peninsula’s natural climate condition. Depending on the seasons the experimental data of the parabolic concentrator productivity have been got in order to obtain heating steam.

Keywords: Oil reservoir engineering, heating by steam, fusion piercing, solar energy plant, parabolic concentrator, parabola through concentrator.

AMS Subject Classification: 83A05, 83C10.

1. INTRODUCTION

For economizing the energy being utilized for oil wells engineering, exploitation and increasing productivity, development of the new methods, modern facilities and equipments application has huge technological advantages because of economical and ecological profits [1, 2]. So, from the economical facet usage of solar and wind energy potentials is profitable. Therefor, before the combined solar and wind power devices application for steam pumping into the wells experimentally, technical and technological parameters of the wells in the oil fields in Absheron peninsula and thermal effect method being demanded should be investigated. Lately the investment for the oil field progress, plants’ remounting for the oil exploration and refining, modern infrastructure and apparatus usage demands technologically perfect results and goods fitting to the latest standards.

That’s why, high-powered solar energy plants use in the majority of the energy consuming processes of the oil industry leads to the reduction in some oil exploration processes cost and ecological clean environment which causes “Green Energy” differentiation. Absheron peninsula possesses rich oil and solar energy potential; therefore in oil extraction processes solar energy facilities exploitation is unavoidable. Initially, oil extraction processes should be mentioned to be multisided and this investigation is for the oil wells which have low productivity, oil extraction coefficient or duration has limited for exploitation in Absheron peninsula. Methods for the purpose of developing these wells’ yield possess both advantages and some deficiencies. Based on the high results, thermal influence is one of the main methods in heating well deep zone and increasing oil extraction coefficient [3].
High powered parabolic and parabolo through concentrators’ usage was considered for the steam heating method application as before mentioned. Suiting to the equipments existing in the oil field condition, technological scheme of the additional solar energy facilities into the current power system was prepared.

Based on the solar energy facilities high temperature obtaining is possible just with the parabolic concentrator. To the account of the rays’ focusing on the middle point of the concentrator more than 30000 C\(^{30000}\) C temperature appears [4] and [5]. Depending on the square the parabolic concentrator may be made by facet type or intact mirrors. The mirrors made by cast glass the accuracy is the highest but their cost is more expensive and fetch some difficulties during transportation and exploitation. In the facet type plates relatively vehement changing of the opening angle reduces the rays’ concentration precision on the same point but such kind of mirrors are too cheap economically and being easily made [6]. Advantage of the parabola through concentrators is both facet mirror production and aluminum leaf polished by electricity or stainless steel films. Concentrator material of the plant made is from country-USA production having \( \delta = 1\) mm and consists of stainless steel plate and solar ray reflecting coefficient equals \( R = 0,91 \). To improve the profitability of the made plants facet type Parabolic, parabola through concentrators with facet mirror and stainless steel plate were provided with sun tracking systems working on to azimuthal and zenithal surfaces [6].

By the purpose of water heating steam obtaining in three plants, correspondingly experiments were carried out in Absheron peninsula’s natural climate condition. Firstly within 2008-2016 periodical measurements of solar energy potential using actinometer, pyranometer, albedometer, and galvanometer were realized and finally solar energy potential of Absheron peninsula and distribution of the potential on the seasons were determined [7, 8]. In order to intensify measurements to obtain too accurate data during 2008-2016 years, digital distance controlling devices possess high accuracy class made in German were used. These equipments work automatic regime and the data got being sent to the computer.

### Table 1. Experimental results of parabolic and parabola through concentrator on the months of the year

<table>
<thead>
<tr>
<th>Months</th>
<th>Total solar radiation amount, ( MC/m^2 )</th>
<th>Average weather temperature, °C</th>
<th>Average wind speed, m/sec</th>
<th>Plant Productivity, kq/m²month</th>
<th>Plant efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>305</td>
<td>3.9</td>
<td>5.5</td>
<td>58</td>
<td>47</td>
</tr>
<tr>
<td>II</td>
<td>370</td>
<td>4.1</td>
<td>5.9</td>
<td>86</td>
<td>66</td>
</tr>
<tr>
<td>III</td>
<td>555</td>
<td>6.3</td>
<td>6.4</td>
<td>128</td>
<td>101</td>
</tr>
<tr>
<td>IV</td>
<td>639</td>
<td>11.2</td>
<td>5.9</td>
<td>360</td>
<td>189</td>
</tr>
<tr>
<td>V</td>
<td>756</td>
<td>17.7</td>
<td>5.7</td>
<td>587</td>
<td>305</td>
</tr>
<tr>
<td>VI</td>
<td>769</td>
<td>22.6</td>
<td>6.1</td>
<td>758</td>
<td>498</td>
</tr>
<tr>
<td>VII</td>
<td>745</td>
<td>25.7</td>
<td>6.7</td>
<td>766</td>
<td>685</td>
</tr>
<tr>
<td>VIII</td>
<td>672</td>
<td>25.7</td>
<td>6.0</td>
<td>702</td>
<td>652</td>
</tr>
<tr>
<td>IX</td>
<td>556</td>
<td>21.8</td>
<td>5.8</td>
<td>655</td>
<td>502</td>
</tr>
<tr>
<td>X</td>
<td>443</td>
<td>16.6</td>
<td>5.5</td>
<td>565</td>
<td>425</td>
</tr>
<tr>
<td>XI</td>
<td>307</td>
<td>11.1</td>
<td>5.2</td>
<td>403</td>
<td>301</td>
</tr>
<tr>
<td>XII</td>
<td>267</td>
<td>6.8</td>
<td>5.2</td>
<td>191</td>
<td>98</td>
</tr>
</tbody>
</table>

For steam heating obtaining purchase, productivity and efficiency of the solar power plants having high and medium temperatures on the months of the year were tabled. While the experiments, According to the climate conditions, total solar radiation amount (\( MC/m^2 \)), average air
temperature ($^\circ$C), average wind speed ($m/sec$) were measured. Productivity and efficiency of parabolic and parabola through concentrators changes depending on the the sun rays intensity with the parabolic curve.

3. Results

To the analysis and experiments in thermal working of oil and natural bitium wells having higher viscosity the highest-powered energy plants are demanded. In order to economize energy and develope the fusion piercing process in the oil field the following results should be taken into considration:

1. New energy equipments supplied with independent work rejime, (without workmen) should be applied;
2. Energy facilities working with free of charge energy source economically to save fuels and electricity are neccessary;
3. Power devices working with clean energy ecologically to reduce the fines by the suitable official organization juristically are certain;
4. The modern autonomous energy devices produced at the local condition aught to be used;
5. The high-efficient power plants for more energy consuming processes to decrease the loss and to develop the quality of the goods have to be utilized.
6. The long-lived energy plants or station should be applied to economize both energy resersves and devices’ cost must be developed.

References

THE METALLICITY OF ATMOSPHERE OF THE POST AGB

HD161796 (F3Ib) STAR

Z.A. Samedov1

1Baku State University, Z. Khalilov str.23, Az 1148, Baku, Azerbaijan
e-mail: zahir.01@mail.ru

1. Introduction

Post AGB stars are out of the asymptotic arm of giant stars, but are still stars that have not yet reached planetary nebulae space. Post AGB phase is the evolutionary phase between AGB and the emergence of planetary nebulae. The duration of this stage is very small: $\approx 10^3$ years. Post AGB stars are the second-class type A, F, G spectroscopic supergiant stars located far beyond the star-forming regions. These stars are different from normal supergiant stars, and they are extraordinarily supergiant stars. The masses of the Post AGB stars are smaller than the masses of normal supergiant stars, and their luminosity is the luminosity of normal giant stars. Post AGB stars are located in the wide galactic width, and their spectrum is observed an infrared radiation excess. This shows the formation of the dust environment on the surface of the stars at the end of the AGB stage. These stars have the chemical composition in which can not explain the theory of modern evolution of stars. The determination of the chemical composition of the post AGB stars is one of the actual problems of astrophysics.

In this work, the metallicity of the atmosphere of the star has been determined. In our future work it is envisaged to determine the abundance of light elements that have evolved in the chemical evolution. The observation material of the star was taken in the 2-meter telescope of the Shamakhi Astrophysical Observatory within the wavelength range $\lambda\lambda3900 \div 6600AA^\circ$, the atlas was constructed and the equivalent width of the spectral lines was measured.

2. Atmosphere parameters: effective temperature, gravity

The effective temperature of the star and the surface gravity are determined by the model method. This method is shown in detail in [6]. The following criteria are used:

1. Comparison of the values of the index $[c_1]$ measured from observation and theoretically calculated. The index $[c_1]$ is determined by the expression $[c_1] = c_1 - 0.2(b - y)$ in the the photometric system $uvby\beta$. This is exempt from the effect of absorption in the quantitative interstellar space. It is desirable to use the index $[c_1]$, when determining the fundamental parameters of stars by the model method.

2. Comparison of the measured from observation and theoretically calculated values of the index $\beta$. This quantity actually measures intensity on the line $H_\beta$, free from the interstellar absorption.

3. Comparison of the measured from observation and theoretically calculated values of the index $Q$.

In the photometric system $UBV$ the index $Q$ is determined by the expression $Q = (U - B) - 0.72(B - V)$. The quantity $Q$ is exempt from the interstellar absorption. The observed values
of the quantities \( [c_1], \beta, Q \) are calculated using the catalog [3]. The theoretical values of the quantities \( [c_1], Q \) are calculated using the [1].

The \( \log g - T_{eff} \) diagram is building on the base of the above criteria (Fig.1).

![Figure 1](image)

Figure 1. A diagram defining the \( T_{eff} \) and \( \log g \) parameters of the HD161796 star.

From this diagram the star’s parameters are defined:

\[
T_{eff} = 6550 \pm 200K, \log g = 0.75 \pm 0.2.
\]

3. MICROTURBULENT VELOCITY, IRON ABUNDANCE ELEMENT

In the atmosphere of the star, microturbulent velocity and iron abundance are determined by the lines \( FeII \). To determine the microturbulent velocity it must be a plurality of lines that contain a wide equivalent widths range of the atoms or ions of any given element. The microturbulent velocity is chosen such that the abundance of elements determined by the different lines does not change with the increasing of the equivalent widths \( W_\lambda \).

As is shown in [7] that the microturbulent velocity \( \xi_t \) increases with the altitude of the height in the the atmosphere of spectral class stars \( F \). The effect is more effective if the line is stronger. For weak lines, this dependence is neglected and it is assumed that the microturbulent velocity \( \xi_t \) is stable in the atmosphere of star. Only the weaker lines are used when determining the microturbulent velocity \( \xi_t \). These lines are formed in deep layers of the atmosphere, these layers are parallel and in LTT form.

The iron abundance is calculated by giving different values to the microturbulent velocity \( \xi_t \) based on the Kurucz model [5] with the parameter. The iron abundance is determined on the basis of comparison of the values measured from observation and theoretically calculated equivalent width of lines \( FeII \). The atomic data of the spectral lines were taken from the database VALD-2 [4]. There is no correlation between \( \log \varepsilon(FeII) \) and \( W_\lambda \) when \( \xi_t = 6.5 km/san \) (Fig.2).

![HD161796](image)

HD161796 (F3ib) \( \xi_t = 6.5 \text{ km/s} \)
Thus, in the atmosphere of star determines the value for the microturbulent velocity \( \xi_t = 6.5 \text{ km/san} \). At the same time, the iron abundance determines too:

\[
\log\varepsilon(Fe) = 7.03 \pm 0.06.
\]

The parameter \([Fe/H] = \Delta\log\varepsilon = \log\varepsilon(Fe) - \log\varepsilon(Fe)\) is called the metallicity indicator of the star. Here \( \log\varepsilon(Fe) \) is the iron abundance in the sun: [8].

4. Conclusion

In the work the atmosphere of the post \( AGB \) \( HD161796 \) (F3Ib) star is considered. In the atmosphere of the star the effective temperature \( T_{\text{eff}} \) and the surface gravity \( g \) are determined:

\[
T_{\text{eff}} = 6550 \pm 200 \text{ K}; \log g = 0.75 \pm 0.2.
\]

Based on the \( Fe\text{II} \) lines the microturbulent velocity \( \xi_t \) is determined: \( \xi_t = 6.5 \text{ km/san} \). In the atmosphere of the star the iron abundance is calculated and compared with the abundance in the Sun. The iron abundance is determined on the basis of comparison of the values measured from observation and theoretically calculated equivalent width of lines \( Fe\text{II} \). The iron abundance is less than the abundance in the Sun \( \log\varepsilon(Fe\text{II}) = 7.03 \pm 0.06 \). To explain this property of the chemical composition, it is assumed that the number of all elements in the post \( AGB \) star atmosphere was originally normal (with solar structure) and then some of atoms of an element of iron group have been used on formation of dust particles as a result of formation of the dust environment on surfaces of these stars.

**Keywords:** Fundamental parameters - stars, individual-\( HD161796 \) (F3Ib), chemical composition stars.

**AMS Subject Classification:** 85Axx.

**References**

1. Introduction

Remote Photoplethysmography (rPPG) is a research field of medical and biometry systems based on extracting subtle changes on human skin in order to obtain heart rate (HR). Due to the blood flow, there occurs some micro-level changes on human skin that can reveal the HR of a person. Since HR contains valuable information about a person’s current state and this method requires no external device with being completely contactless, this topic is widely studied by researchers. Current researches on rPPG are mostly performed in six steps: face detection, region of interest (ROI) creation, color variation analysis, raw signal extraction, signal filtering and HR estimation, respectively. Studies have shown that HR can be measured with high accuracy using rPPG method under adequate ambient light with standard cameras. Since these measurements are made with evaluating very small values, the signal quality may not always be as good as expected due to some noises caused by different sources like camera noise and motion noise. Converting the signal from time domain to frequency domain mostly results in accurate HR measurement regardless of signal quality. However, in some cases it may be necessary to obtain a better quality signal. For example, most of the proposed rPPG methods still calculate a HR value when a non-live face like printed photo of a face is shown to the camera. They presume the signal belongs to a real face and calculates a HR value even if the produced rPPG signal is not as expected. When it comes to distinguish a rPPG signal whether it belongs to a real face or not, acquired raw signal must be calculated and processed with more accurate methods.

In this paper, a new approach is proposed to improve received rPPG signal quality. ROI selection is designed to be more stable and a mask is applied to the ROI to get rid of noises. This enhanced rPPG signal can be used to get more robust and coherent results.

2. Proposed method

A basic HR measurement system consists of steps including face detection, ROI creation and intensity calculation, signal generating, signal filtering and HR measurement. The contribution of this study is related to the ROI creation and intensity calculation where an approach to improve rPPG signal quality is proposed. This approach utilizes some image processing methods to make ROI creation more stable and to make intensity calculation more precise in favor of improving the quality and trustworthiness of the raw rPPG signal.

In the first step, a face detector is applied to image to locate face region. Once the face region has been detected, a suitable ROI should be selected to be analyzed. Existing studies have shown that forehead and cheek regions are the most sensitive regions of face to the changes
Figure 1. Forehead ROI selection: (a) 68 face landmark; (b) Using landmarks 39 and 42 to find forehead region; (c) Placing the ROI considering the angle

as a result of blood flow [3]. Considering observability, forehead region is preferred as being more favorable for analysis. Aforementioned studies prefer getting the ROI region by a simple calculation from face region, but as HR is a very sensitive measurement, this step must be handled more delicately to locate more stable ROI. A more stationary forehead region selection is proposed by getting more distinctive points on face region which are called face landmarks [2]. A face has 68 landmarks to represent basic important points (Fig.1 (a)).

Landmarks are used to represent specific regions of face like eyes, mouth and nose. Even a small shift on face region can affect defining the same ROI region negatively. For this reason, using face landmarks can help making more successful determinations by referring to these fixed positions. As shown in Fig.1 (b), forehead can be calculated from rightmost point of left eye and leftmost point of right eye. Deducing from the landmarks 39 and 42 to forehead region is a consistent way of ROI creation as these points are stationary on face. As shown in Fig.1 (b), distance between points 39 and 42 is the width of the ROI, height is the half of width and position is determined as above from these points for one and a half times of width of ROI in length. In case of rotation on face (Fig.1 (c)), related ROI is calculated by using the rotation angle so the whole process results in taking almost the same region into account.

Another contribution of this study is to apply a mask to the forehead region to eliminate indecent parts. Aforementioned studies on HR extraction analyzed different color spaces and channels and it is shown that green channel of RGB color space provided the strongest rPPG signal since absorption of hemoglobin is most sensitive to green light [8]. As this information is also verified by our own experiments, the green channel is preferred.

Most of the studies preferred just calculating the average value of all pixels in forehead region to get the rPPG signal [1, 4]. It was noticed that with this approach, some artifacts and noises in forehead region affect rPPG signal negatively. In-depth analysis showed that there are mainly four kinds of noise and artifacts which may distort signal accuracy: motion change, illumination change, camera-induced noise and video codec artifacts. In order to eliminate these unwanted factors, a mask to apply to the ROI is proposed. Proposed mask includes four steps (Fig.2). Firstly, pixels that have abnormally changed in current frame relative to the previous frame are detected with frame differencing and this difference matrix is thresholded (Fig.2 (a, b)). Otsu, which is preferred here as thresholding method is a well-known thresholding method that automatically finds the threshold value by analyzing the image in order to divide the image into two intensity levels by minimizing the weighted within-class variance and maximizing between-class variance [5, 6]. With using Otsu, a binary image called mask is created as abnormal intensity changes have been discarded. As seen in Fig.2 (b), video codec compression artifacts and motion noises are detected and represented with white. These pixels are discarded in order to keep the signal not to be interfered.
In addition to this thresholding step, histogram of the masked frame is calculated and other abnormal pixels are eliminated by histogram analysis. Since a forehead region is expected to be consisting of nearly the same intensity level of pixels, the forehead region intensity distribution is concentrated in approximately the same region of histogram, as shown in Fig.2 (c). Thus, excluding any part of ROI whose intensity level is not close to the peak of histogram does not impair integrity of initial rPPG value, but makes it more truthful as they may be pixels contrary to the integrity. A histogram of skin is usually concentrated around the same bin. In this case, it is a sufficient operation to eliminate contradictory pixels by selecting an area of histogram surrounding the peak bin. This is done by selecting a window of \( n \) bins which covers \((n-1)/2\) bins from right and \((n-1)/2\) bins from left of the peak bin of histogram. Having \( n \) large causes the method not effective enough and having it narrow causes the rPPG signal to be distorted. With such a forehead image, it is seen that more than 75 percent of histogram is covered and abnormal pixels are eliminated when \( n = 11 \). As shown in Fig.2 (d), by using the masking operation, pixels which are not similar to the overall of the histogram is eliminated and a more trustworthy rPPG signal is calculated by the average of the rest of pixels fulfilling the conditions.

Fig.3 shows the difference of obtained signal quality between filtered and non-filtered ROI. A video from CASIA video database [9] is analyzed for 125 frames with both methods; the first one takes average of whole ROI as rPPG signal value and the second one uses the proposed approach. The difference is obviously seen after a simple zero-phase filtering is applied to the raw rPPG signals.

3. Conclusion

rPPG is a promising technique to reveal hidden vital signs of human skin like heart rate. As it requires no additional device and being totally non-contact, it may be useful in some cases like patient monitoring and liveness verification. However, rPPG signal quality is affected by ease
with many factors like variant illumination and motion of subject. This paper introduces a novel method to improve the quality of rPPG signal to overcome such problems. This method filters the ROI from pixels inharmonious from the majority and it results in a more stable and robust rPPG signal. Beside the output of this study may be useful to make a HR system more accurate, it may play a crucial role on a HR based liveness detection system. The future works will be directed toward the improvement of filtering phase and applying it on a liveness detection system.

**Keywords:** Remote Photoplethysmography, heart rate measurement, signal noise reduction.

**AMS Subject Classification:** 68U10, 60G35.

**References**


REAL-TIME VIDEO SYNOPSIS FOR SURVEILLANCE CAMERAS

REFIK SAMET¹, KEMAL BATUHAN BASKURT¹

¹Department of Computer Engineering, Ankara University
Ankara Univ. 50. Yil Kampusu, 06830 Golbasi/Ankara, Turkey
e-mail: samet@eng.ankara.edu.tr, batuhanbaskurt@gmail.com

1. INTRODUCTION

Video synopsis is an activity-based video condensation approach to achieve efficient video browsing and retrieval for surveillance cameras. An activity represents collection of continuous instances belonging to same object in video frames. Activities extracted from the source are shifted in the time domain to find their optimal positions with the minimum number of collisions. Activities from different time periods can be shifted into the same frame through pixel-based analysis. Therefore, more efficient condensation performance is achieved compared to frame-based video summarization methods.

Video synopsis generation starts with object detection, then object tracking is applied to create activities. Next, optimization of the selected activities is applied to obtain optimal temporal rearrangement. Afterwards, a time-lapse background representing the time period of the selected activities is created, and finally, activities are stitched to the generated background. Optimization is the most important step in video synopsis. All optimization methods aim to obtain mapping of activities from the source video to proper positions in the synopsis video. The final goal is to display all of the activities in the shortest time period while avoiding collisions as much as possible. Generally, the optimization problem is defined as minimization of a global energy function that consists of several costs such as maximum activity, background and temporal consistency and activity collision [3, 4].

In this paper, an online optimization approach that can be applied in real-time to the surveillance cameras is proposed. A novel grid map representation is proposed to handle the bottleneck on iterative calculation of activity collision in the optimization phase. Through the run-time performance gained by proposed effective grid map representation, an extensive energy function covering activity, collision and chronological disorder costs is defined. Therefore, precision of the optimization is increased while the real-time performance is preserved.

2. THE PROPOSED APPROACH

An activity (A) represents the collection of object instances that are being tracked in the scene. Each instance is represented by its pixel coordinate (x, y) in the corresponding video frame. Therefore an activity is formulated as A = (x₁, y₁), (x₂, y₂), ..., (xₙ, yₙ), where n represents the total number of frames that the activity is visible. Collision between the activities is calculated by comparing coordinates of the instances with each other. This operation is computationally expensive considering that it is applied for all possible temporal shifts to find the optimal one.

Grid representation is proposed to decrease complexity of the collision control and gain run-time performance in order to define an efficient energy function. Input image is divided into
MxN grids, then each activity instance is represented by the ID of the grid occupied in corresponding video frame. In this way, an activity instance can be represented by 1-byte grid ID, through limiting number of grids to fit in, instead of pixel coordinates. Therefore, the collision control and temporal shifting of the activities are performed by bitwise XOR and shift operations as illustrated in Figure 2.

(a) Pixel representation  (b) Grid representation

Figure 1. Activity collision control of pixel and grid representation

A grid map that contains grid representation of all the activities \(A_s\) being displayed in synopsis video is also defined. Mentioned grid map is built efficiently by bitwise OR operation of the activities as illustrated in Figure 2. Candidate activities \(A_c\) that will be located into \(A_s\) are compared with grid map instead of comparing with the each activity separately. In this way, collision comparison is accelerated to gain more run-time performance to spend on energy minimization phase.

Figure 2. Grid map

An energy function \(E\) consisting of activity \((E_a)\), chronological disorder \((E_c)\) and occlusion \((E_o)\) costs is defined as follows:

\[
E = \sum_{i \in A_c} E_a(i) + \sum_{i \in A_c, j \in A_s} [\alpha E_c(i, j) + \beta E_o(i, j)],
\]

where \(A_s\) is synopsis activities whose optimal temporal position has already been calculated. These are the activities that are being displayed or waiting for the starting frame in the optimized activity set. \(A_c\) represents the candidate activities that are currently detected. \(\alpha\) and \(\beta\) are weight parameters of the \(E_c\) and the \(E_o\) respectively.

Proposed approach uses motion blobs to detect objects in the scene [1]. Activities are created by tracking detected motion blobs using correlation filter based object tracking method [2]. Thereafter, energy function is calculated iteratively for each position of the grid map once a new activity is detected. Optimal position of the new activity is determined by finding the temporal position with minimum energy.

3. Experimental results

Experiments of the proposed approach were performed on a public dataset that have 320x240 pixels resolution, 20 minutes duration with 20 fps frame rate and 235 activities in total [5]. Experiments were performed on a computer with Intel Xeon Quad Core 3.40 GHz CPU and 8 GB memory.
Performance of the proposed approach is compared in following two performance metrics: overlap ratio (OR) and chronological disorder ratio (CDR) where both of them take value between 0 and 1. OR is used to measure occlusion ratio of the activities and calculated as follows:

\[
OR = \frac{1}{W \cdot H \cdot T_s} \sum_{t=1}^{T_s} \sum_{x=1}^{W} \sum_{y=1}^{H} \left\{ \begin{array}{ll} 1, & p(x, y, t) \in \text{Collision}, \\
0, & \text{otherwise}, \end{array} \right.
\]

where \(W\) and \(H\) are width and height of synopsis video frame, respectively. \(T_s\) represents the number of frames in synopsis video. \(p(x, y, t)\) represents a pixel of the frame \(t\) of synopsis video and \(\text{Collision}\) represents the collision region of the activities. The smaller the OR is the fewer occlusion of the activities that increases visual quality of the synopsis. CDR indicates ratio of the activities displayed chronologically disordered in synopsis video and calculated as follows:

\[
CDR = \frac{A_d}{A_s},
\]

where \(A_d\) and \(A_s\) are number of disordered and total activities in synopsis video, respectively. The smaller the CDR is the fewer violation of the chronological order of the activities.

Proposed approach is compared with JVS [5] which is one of the most recent and successful approach. Table 1 shows minimum and maximum results of the metrics according to different parameter configurations of the studies. Proposed approach shows 17.6 times better result on OR and 2.25 times better on CDR.

### Table 1. Results of comparison

<table>
<thead>
<tr>
<th>Method</th>
<th>OR</th>
<th>CDR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Min</td>
<td>Max</td>
</tr>
<tr>
<td>JVS [5]</td>
<td>0.009</td>
<td>0.0175</td>
</tr>
<tr>
<td>Proposed approach</td>
<td>0.00051</td>
<td>0.00074</td>
</tr>
</tbody>
</table>

In this paper, a novel online activity optimization approach has been proposed for video synopsis. Proposed approach is built on grid map representation which provides opportunity to perform activity collision and temporal shifting by bitwise operations efficiently. Therefore, runtime performance of the optimization phase is improved significantly. In this way, an extensive energy function that gives equal weight to the activities is defined to obtain better optimization results comparing the existing online activity optimization approaches.

**Keywords:** Online activity optimization, video synopsis, video condensation, real-time systems.

**AMS Subject Classification:** 68U10, 65D19.

**REFERENCES**


MODELING CURVES BY A QUADRATIC TRIGONOMETRIC B-SPLINE
WITH POINT SHAPE CONTROL

M. SARFRAZ¹, S. SAMREEN², M.Z. HUSSAIN³

¹Department of Information Science, College of Computing Sciences and Engineering, Kuwait University, Kuwait
²Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan
³Department of Mathematics, University of the Punjab, Lahore, Pakistan
e-mail: malikzawwar.math@pu.edu.pk, prof.m.sarfraz@gmail.com, shamailasamreen16@gmail.com

1. INTRODUCTION

In this paper, it is established to build a Nu B-spline like basis using quadratic trigonometric functions with the view of its applications in curve modeling for Computer Aided Geometric Design (CAGD), geometric modeling, and computer graphics along with well meticulous shape impacts of parameters. The proposed scheme has several features and benefits together with the followings:

- The scheme maintains the suitable geometric properties of B-splines.
- The scheme accomplishes the \( GC^2 \) geometrical smoothness.
- It owns good features of trigonometric splines.
- The scheme is sanctified with interesting shape designing features.
- In this method, the B-spline like curve representation is put forward for consideration by a transformed Bezier form.
- The proposed method recovers a \( C^2 \) quadratic trigonometric B-spline.
- The method has one family of shape parameters, which contribute to a different shape effects with point tension, interval tension and global tension.

2. QUADRATIC TRIGONOMETRIC NU SPLINE

Let \( F_i \in \mathbb{R}^n \) be given values at knots vectors \( t_i, i = 0, 1, \ldots, n - 1 \), where \( t_0 < t_1 < \ldots, < t_n \).

The quadratic trigonometric function at these knots, is defined for \( t \in [t_i, t_{i+1}], i = 0, 1, \ldots, n-1 \), by

\[
S_i(t) = \sum_{j=0}^{3} A_j(\vartheta)P_j,
\]

where

\( A_j(\vartheta), j = 0, 1, 2, 3, \ldots \) are quadratic trigonometric basis functions defined as: \( A_0(\vartheta) = (1 - \sin \vartheta)^2, A_1(\vartheta) = 2(1 - \sin \vartheta) \sin \vartheta, A_2(\vartheta) = 2(1 - \cos \vartheta) \cos \vartheta, A_3(\vartheta) = (1 - \cos \vartheta)^2 \), with \( 0 \leq \vartheta \leq \frac{\pi}{2}, \vartheta(t) = \left( \frac{t-t_i}{h_i} \right) \frac{\pi}{2}, h_i = t_{i+1} - t_i, i = 0, 1, \ldots, n-1 \).

Also \( P_j \in \mathbb{R}^n, j = 0, 1, 2, 3, \ldots \) are the control points defined as:

\[
P_0 = F_1, P_1 = V_i = F_i + \frac{h_i}{\pi} M_i,
\]

\[
P_2 = W_i = F_{i+1} - \frac{h_i}{\pi} M_{i+1}, P_3 = F_{i+1}.
\]
Moreover, these functions are Bezier weight functions with
\[ \sum_{j=0}^{3} A_j(\vartheta) = 1. \] (3)

The following interpolation properties are satisfied by piecewise quadratic trigonometric defined in (1),
\[ S(t_i) = F_i, S(t_{i+1}) = F_{i+1}, \] (4)
\[ S'(t_i) = M_i, S'(t_{i+1}) = M_{i+1}, \] (5)
and it behaves like a Hermite. Now, by inserting the Nu spline characteristics into the quadratic trigonometric spline, GC^2

Nu spline interpolates
\[ f(t_i) = F_i, i = 1, 2, \ldots, n \]
and we have the second continuity at the joints of the curve segments as in (5).
\[ S''(t_i) = \nu_i S''_{i-1}(t_i) + S''_{i+1}(t_i), \] (6)

In this way, an interpolatory Nu spline is shaped to form a system of consistency equations (7) in unknowns \( M_i \)'s using (2), (4), (5) and (6):
\[ \frac{\pi}{2h_{i-1}} D_{i-1} + \left( \frac{\pi}{h_i} + \nu_i + \frac{\pi}{h_{i-1}} \right) D_i + \frac{\pi}{2h_i} D_{i+1} = \frac{\pi^2}{2h_i} \delta_i + \frac{\pi^2}{2h_{i-1}} \delta_{i-1}, \] (7)
where
\[ \delta_i = \frac{F_{i+1} - F_i}{h_i}. \] (8)

For appropriate end conditions on the shape parameters. \( \nu_i \geq 0, \forall i \), the above tri-diagonal linear system, for derivatives \( M_i \)'s, is diagonally dominant.

3. Structure of freeform basis

To build the B-spline like basis functions for the proposed spline in Section 2, let us construct a local function \( \omega_j(t) \). This function is such that additional knots on both sides of the interval \([t_0, t_n]\) are introduced by:
\[ t_{-3} < t_{-2} < t_{-1} < t_0 \quad \text{and} \quad t_n < t_{n+1} < t_{n+2} < t_{n+3}. \] (9)

The function \( \omega_j(t), j = -1, \ldots, n + 2, \) is defined as follows:
\[ \omega_j(t) = \begin{cases} 0, & \text{if } t < t_{j-2}; \\ 1, & \text{if } t \geq t_{j+1}; \end{cases} \] (10)

While, on the remaining three intervals \([t_i, t_{i+1}], i = j - 2, j - 1, j, \) \( \omega_j(t) \) has the quadratic trigonometric form similar to the one defined by (1):
\[ \omega_j(t) = A_0(\vartheta) \hat{F}_{j,i} + A_1(\vartheta) \hat{V}_{j,i} + A_2(\vartheta) \hat{W}_{j,i} + A_3(\vartheta) \hat{F}_{j,i+1}. \] (11)

Here \( A_k(\vartheta), k = 0, \ldots, 3, \) are defined similar as in (3), also
\[ \hat{F}_{j,i} = \omega_j(t_i), \hat{V}_{j,i} = \omega_j(t_i) + \frac{h_i}{\pi}, \omega_j'(t_i), \hat{W}_{j,i} = \omega_j(t_{i+1}) - \frac{h_i}{\pi} \omega_j'(t_{i+1}) \] (12)

Now, by applying the constraints \( \nu_i \geq 0, \forall i \) on \( \omega_j(t) \), we have obtained the following:
\[ \omega_j(t_{j-2}) = 0, \omega_j'(t_{j-2}) = 0, \omega_j''(t_{j-2}) = 0, \]
\[ \omega_j(t_{j-1}) = \alpha_{j-1}, \omega_j'(t_{j-1}) = \hat{\alpha}_{j-1}, \]
\[ \omega_j(t_j) = 1 - \beta_j, \omega_j'(t_j) = \hat{\beta}_j \] (13)
Now, by taking the difference of two consecutive \( \omega_j(t) \) functions, the free form quadratic trigonometric Nu B-spline (QTNBS) basis are determined as: 
\[
B_j(t) = \omega_j(t) - \omega_{j+1}(t), \quad j = -1, \ldots, n+1.
\]
Thus for any \([t_i, t_{i+1}]\) if
\[
B_j(t) = A_0(\vartheta)F_{j,i} + A_1(\vartheta)V_{j,i} + A_2(\vartheta)W_{j,i} + A_3(\vartheta)F_{j,i+1},
\]
where \( F_{j,i} = B_j(t_i), \quad V_{j,i} = B_j(t_i) + \frac{h_i}{2} B'_j(t_i), \quad W_{j,i} = B_j(t_{i+1}) - \frac{h_i}{2} B'_j(t_{i+1}), \)

**Proposition 1.** The QTNBS basis functions have the geometric properties as follows:
- **Local support:** \( B_j(t) = 0, \) for \( t \notin (t_{j-2}, t_{j+1}) \).
- **Partition of unity:** \( \sum_{j=-1}^{n+1} B_j(t) = 1 \) for \( t \in [t_0, t_n] \).
- **Positivity:** \( B_j(t) \geq 0 \) , for all \( t \).

4. **Freeform QTNBS Curve Design**

For free form curve modeling, it is essential to figure the curve depiction as follows:
\[
S(t) = \sum_{j=-1}^{n+1} P_j B_j(t) \text{ for } t \in [t_0, t_n],
\]
with \( P_j \in \mathbb{R}^n \) as the control points. Now by using Proposition 1, the Equation (15) can be re-written as:
\[
S(t) = \sum_{j=-1}^{i+2} P_j B_j(t), \quad t \in [t_i, t_{i+1}], \quad i = 0, 1, \ldots, n-1.
\]
After further operations, (16) transforms to the piecewise B?zier form acquired as follows:
\[
S(t) = H_0(\vartheta)F_i + H_1(\vartheta)V_i + H_2(\vartheta)W_i + H_3(\vartheta)F_{i+1},
\]
where, for computational purpose, one can certainly use transformed Bezier form.

5. **Geometric properties**

The geometric properties are given by the following propositions:
- The QTNBS curve entirely lies within the convex hull determined by its control points \( \{F_i, V_i, W_i, F_{i+1}\} \).
- Let \( S(t) = \sum_{j=0}^{3} P_j B_j(t) \) for \( t \in [t_i, t_{i+1}] \) be a QTNBS curve with control points \( P_j = \{F_i, V_i, W_i, F_{i+1}\} \in \mathbb{R}^n \). Then any hyper plane of dimension \( N-1 \), will cross the QTNBS curve no more times it crosses the control polygon P joining the control points.
- The QTNBS curve \( S(t) = \sum_{j=0}^{3} P_j B_j(t) \) for \( t \in [t_i, t_{i+1}] \) with control points \( P_j = \{F_i, V_i, W_i, F_{i+1}\} \in \mathbb{R}^n \) is invariant under affine transformation.
- Let \( \nu_i \geq 0, i = j - 2, \ldots, j + 1 \). Then QTNBS converges uniformly to linear polynomial B-spline.
- Let \( \nu_k \) for some \( k \in 1, 2, \ldots, n - 1 \), then \( \lim_{\nu_k \rightarrow \infty} S(t) = P_k \) determines point tension property which ultimately leads to Interval and Global Tension properties.

**Keywords:** Geometric modeling, spline, Bezier curve, geometric continuity.

**AMS Subject Classification:** 42A05, 42A10, 65D07, 65D10, 65D17.

**References**


ON THE INVERSE SPECTRAL PROBLEM IN STURM-LIOUVILLE OPERATOR HAVING SPECIAL SINGULARITY

MURAT SAT$^1$, ETIBAR S. PANAKHOV$^2$, MEHMET KAYALAR$^3$

$^1$Erzincan University, Department of mathematics, 
$^2$Institute of Applied Mathematics, Baku State University, Azerbaijan 
$^3$Erzincan University, Vacoatalnai high school

e-mail: murat_sat24@hotmail.com, epenahov@hotmail.com mehmetkayalar24@hotmail.com

ABSTRACT. The aim of this paper is to solve the wellposedness problem for the Sturm-Liouville operator with Coulomb potential. By using the Mizutani’s method, it will be shown that the difference potential functions $|\tilde{q}(x) - q(x)|$ is small, if $\{\mu_n, \alpha_n\}$ are close to $\{\lambda_n, \alpha_n\}$ $(n \in \mathbb{N})$ in a certain sense.

Keywords: Eigenvalue, Coulomb potential, inverse problem, norming constants.

AMS Subject Classification: 34A55, 47E05, 34L05.

1. Introduction

In spectral theory, the inverse problem is the usual name for any problem in which it is necessary to ascertain the spectral data that will determine a different operator uniquely and a method of construction for this operator from the data. A problem of this kind was first started and investigated by Ambartsumyan in 1929 [1]. Since 1946, various forms of the inverse problem have been studied by numerous authors Borg [5], Levinson [12], Levitan [13], etc. and now there exists an extensive literature on the question [4, 6, 9, 10]. Later, the inverse problems having specified singularities were considered by a number of authors [2, 3, 7, 8, 11, 15 – 23].

Some integral representations have been taken in [21, 22]. In particular, the integral representation for special functions which are given in those papers have been got by using Fourier transformations. Moreover, this kind of representation is useful for investigating the spectral functions of singular Dirac and Sturm-Liouville differential operators in half-and all axis, respectively. Spectral functions are important for determining the operators, that is, for solving the inverse problem for differential equations which generate the operator with initial conditions are more useful for investigating the spectral properties of this operator.

In case $q(x) \equiv 0$, since this operator is the singular Sturm-Liouville operator with Coulomb potential, linearly independent solutions of this kind of differential equation could be given with hypergeometric functions and this integral representation is also a representation for hypergeometric functions. For this reason, obtaining, this kind of integral representation is so important. Therefore, when obtained, these integral representations can be used for asymptotic behaviours of hypergeometric functions as $x \to +\infty$.

In this paper, we are concerned with the wellposedness problem for Sturm-Liouville operator having special singularity, using Mizutani’s method [14].

Firstly, we give a brief knowledge of the Sturm-Liouville equation with coulomb potential.
Learning about the motion of electrons moving under Coulomb potential is of remarkable interest in quantum theory. Solving these types of problems provides us to determine energy levels not only hydrogen atom but also single valence electron atoms such as sodium.

For hydrogen atom, Coulomb potential is taken by $U = \frac{-e^2}{r}$, where $r$ is the radius of the nucleus, $e$ is electronic charge. Accordingly, we use time dependent Schrödinger equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + U(x, y, z)\Psi, \quad \int \Psi^2 dxdydz = 1,$$

where $\Psi$ is the wave function, $\hbar$ is Planck’s constant and $m$ is the mass of electron. If we make the necessary transformation, then we can have a Sturm-Liouville equation with Coulomb potential

$$-y'' + \left[\frac{A}{x} + q(x)\right]y = \lambda y,$$

where $\lambda$ is a parameter which corresponds to the energy and $A$ is constant [4].

Now let us consider $\sum_{n=0}^{\infty} \sqrt{\lambda_n} j_n(\tilde{\alpha}_n) \varphi_n(x) \beta_n(x) = \lambda_n \beta_n(x) + j_n(\mu_n) \eta_n(x) + O(\ln n n^2)$ as a spectrum of the following singular Sturm-Liouville problem

$$-y'' + \left[\frac{A}{x} + q(x)\right]y = \lambda y, \quad \lambda = s^2, \quad 0 < x \leq \pi,$$

$$y(0) = 0,$$

$$y'(\pi, \lambda) - Hy(\pi, \lambda) = 0,$$

where $q(x) \in L^1[0, \pi]$, $H$ is finite number and $\varphi(x) \in C[0, \pi]$.

Let us denote by

$$\varphi(x, s) = \sin sx + \int_0^x s\sin(s(x-t)) \left[\frac{A}{t} + q(t)\right] \varphi(t, s) dt,$$

the solution of the equation (1) satisfying the initial conditions

$$\varphi(0, s) = 0 \quad \text{and} \quad \varphi'(0, s) = s,$$

where $\varphi(x, s) \in C[0, \pi]$. Eigenvalues of the problem (1)-(3) are roots of the (3). These spectral characteristics and and eigenfunctions satisfy the following asymptotic expression, respectively [2]:

$$s_n = \sqrt{\lambda_n} = n + \frac{1}{2} + \frac{A \ln(n+\frac{1}{2})}{2\pi} + \frac{c_0}{(n+\frac{1}{2})} + O\left(\frac{\ln n}{n^2}\right),$$

$$\alpha_n = \frac{\pi}{2} + \frac{A\pi^2}{4} \frac{1}{(n+\frac{1}{2})} + O\left(\frac{\ln n}{n^2}\right),$$

$$\varphi(x, \lambda_n) = \varphi_n(x) = \sin(n+\frac{1}{2})x + \frac{A \ln(n+\frac{1}{2})}{2\pi} \cos(n+\frac{1}{2})x + \frac{A\pi}{4} \sin(n+\frac{1}{2})x$$

$$-\frac{\cos(n+\frac{1}{2})x}{(n+\frac{1}{2})} \beta(x) - \frac{A}{2} \cos(n+\frac{1}{2})x \ln(n+\frac{1}{2})x + O(\frac{\ln n}{n^2}),$$

Theorem 1. If

$$B \equiv \sum_{n=0}^{\infty} \left\{ \sqrt{\lambda_n} |\tilde{\alpha}_n - \alpha_n| + |\mu_n - \lambda_n| \right\}$$

is sufficiently small, then we have
where $C_1 > 0$ is a constant depending only on $q(x)$ and $H$.

References

IMPACTS OF THE DEVALUATION ON AZERBAIJAN EXPORT*

E.R. SHAFIZADEH1, R.M. ALIYEV2, V.A. BAYRAMOV2, J.P. HUSEYNOV2

1Baku State University, ANAS Institute of Control Systems, Baku, Azerbaijan
2ANAS Institute of Economy, Baku, Azerbaijan
e-mail: elnure.sh@mail.ru, ramil-turkel@mail.ru, ceyhun_huseynov@hotmail.com

1. INTRODUCTION

Stability of the national economy is one of the main conditions and may be a primary one for ensuring macroeconomic stability in the Republic of Azerbaijan. Sustainability of the national economy is one of the main characteristics of the national economy as a single system, which reflects its state where aggregate factors of economic, social, political, foreign economic nature enable realization of reproduction periods in the long view. In other words, changes in these factors do not cause disturbance of equilibrium of the national economy. Such sustainability reflects solidity and reliability of elements of an economy and structural and economic ties within a system and durability of the system to internal and external impacts.

Taking into account the abovementioned it can be argued that choosing an optimum currency rate is very important to ensure sustainability of the economy and macroeconomic stability in our country. Numeracy of theoretical views on monetary policy and mainly currency rates shows that the content of the issue is complex and comprehensive. These theories from various aspects study formation of currency rates, their impacts on foreign economic relations and generally on the country’s economic development. Any theory characterizing currency rate and monetary and financial policy cannot serve as a factor forming overall monetary policy of a country. Monetary and financial policy of a country is conducted according to diverse provisions of different theories taking into account the characteristics of general economic policy, regulation of currency rates and level of exchange control. However despite that there’s a common idea in all previous theories on currency rate that currency rates policy of a country has to some extent impact on the volume of import and export, monetary reserves and foreign debts, domestic cash flow and inflation. Despite the fact that impacts of the devaluation on macroeconomic indicators were analyzed by various researchers in different periods, there is no common idea in scientific literature about regarding the devaluation as a stabilizing factor of economic growth. International experience shows that some countries performing quick devaluation can experience economic growth by achieving macroeconomic stability. But in some countries the devaluation aggravated the economic crisis.

It should be underlined that there’s no theory defining optimum devaluation level to ensure macroeconomic stability. From this point of view, it is important to theoretically analyze the interrelation of devaluation and macroeconomic stability.

In today’s globalizing world instability in financial markets aggravated, shrinkage of economic growth in the developed countries and drops in oil prices against increasing global risk became frequent, depreciation of some currencies, including those of raw material exporters intensified.

*This work was supored by Science Development Foundation under the President of the Republic of Azerbaijan-Grant N EIF-KEPTL-2-2015-1(25)-56/56/5.
Table 1. GDP, oil price per barrel and export volume of the Republic of Azerbaijan in 2005-2016 [1]

<table>
<thead>
<tr>
<th>Years</th>
<th>GDP, thousand man.</th>
<th>Oil price per barrel, man.</th>
<th>Export, thousand man.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2005</td>
<td>12522500</td>
<td>54.18115</td>
<td>7881214.5</td>
</tr>
<tr>
<td>2006</td>
<td>18746200</td>
<td>59.82206</td>
<td>12467410.9</td>
</tr>
<tr>
<td>2007</td>
<td>28360500</td>
<td>64.30601</td>
<td>19322275.3</td>
</tr>
<tr>
<td>2008</td>
<td>40137200</td>
<td>80.99333</td>
<td>26400700</td>
</tr>
<tr>
<td>2009</td>
<td>35601500</td>
<td>51.3725</td>
<td>18383100</td>
</tr>
<tr>
<td>2010</td>
<td>42465000</td>
<td>64.19195</td>
<td>23060500</td>
</tr>
<tr>
<td>2011</td>
<td>52082000</td>
<td>76.6009</td>
<td>29388300</td>
</tr>
<tr>
<td>2012</td>
<td>54743700</td>
<td>73.88568</td>
<td>29000300</td>
</tr>
<tr>
<td>2013</td>
<td>58182000</td>
<td>76.5672</td>
<td>28117700</td>
</tr>
<tr>
<td>2014</td>
<td>59014100</td>
<td>71.55297</td>
<td>25530000</td>
</tr>
<tr>
<td>2015</td>
<td>54380000</td>
<td>50.58673</td>
<td>20552800</td>
</tr>
<tr>
<td>2016</td>
<td>60393600</td>
<td>70.95371</td>
<td>28054000</td>
</tr>
</tbody>
</table>

Such complicated economic processes have negative impacts on the Azerbaijan economy which is integrated into the global economy. The sharp fall in oil prices has significantly shrunk country’s foreign currency incomings and balance of payments. Decrease in foreign currency incomings of the country as a result of the fall in global market prices of oil shrunk foreign currency supply in the exchange market and thus, inevitably devaluated the Azerbaijan Manat.

It’s reasonable firstly to consider methodology of economic elements affecting macroeconomic stability. Therefore, mainly the following economic indicators should be constantly monitored:

1. GDP standards;
2. Inflation;
3. Cash flow;
4. Balance of payments;
5. Promotion of export.

2. Econometric model of the Azerbaijan export depending on devaluation

As known, Gross Domestic Product (GDP) reflects the whole final goods and services produced within a certain timeframe in a country. Therefore, Diagram 1 aims to show impacts of macroeconomic stability on GDP in the context of devaluation.

World-known economists also noted in their own papers the possibility of analyzing the economic indicators of any states based on GDP and used various formulas to make sound sense of economic activity with limited resources. In line with the experience of the developed economies, expenditure cross-section of the GDP is divided into four integral parts:

- GDP ($Y$);
- Consumption ($C$);
- Investments ($I$);
- Government order ($G$);
- Net exports ($NX$).

We could form the following equation using the abovementioned economic indicators.

\[
Y = C + I + G + NX.
\]

This equation is in fact an equality which is true with any values of the variables in the equation. That is because every dollar calculated in GDP reflects one of the four kinds of expenditures. Therefore, the analysis of the macroeconomic stability in the Republic of Azerbaijan has been based on the volume of the GDP.
Let us construct a model of dependence of export volume of Azerbaijan on oil prices and the volume of GDP. Therefore, let us enter statistical figures of the export volume, oil prices and the volume of GDP for 2005-2016 into Eviews software packet.

These indicators were entered into Eviews software packet and linear dependence of export volume on oil prices and GDP analyzed. The results of this analysis are shown in the following table. (See Table 2):

As seen from table 2 $R^2 = 0.955$ and sufficiency indicators are good enough. So, the model which we created based on these statistical figures can be assumed as 5% of importance. This model is formulated as follows:

$$Y = 272515.55 * X1 + 0.142 * X2 + [AR(1) = 0.587869723346]$$  \hspace{1cm} (1)

Where, $Y$-export volumes, $X1$-oil price per barrel, $X2$ GDP. The outcomes of the model (1) are as follows:

1. Export volume increases 272515.55 units against the rise of the oil price by 1 unit;
2. Export volume increases 0.142 units against the increase in GDP volume by 1 unit.

**Keywords:** Devaluation, export, GDP, econometric model.

**AMS Subject Classification:** 62P20.
OPTIMIZATION METHODS IN THE STABILITY INVESTIGATION OF
REGULATOR SYSTEMS

A.V. SHATYRKO1, D.YA. KHUSAINOV2

1Ph.D. in Math., Ass.Prof., Brno University of Technology, Brno, Czech Republic
2Dr.Sc, Prof., Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
e-mail: shatyrko.a@gmail.com, d.y.khusainov@gmail.com

ABSTRACT. As a rule, stability conditions for solutions of dynamical systems, obtained by using the second Lyapunov method, have sufficient character. We are looking for a continuously differentiable function, which have to be positive definite, and its total derivative by the system solution is negative definite. Because basically the Lyapunov function is constructed as quadratic form, conditions for sign-definite nature of the function and its total derivative become conditions for the positive definiteness of certain special matrices.

Keywords: Absolute stability, direct Lyapunov method, convex optimization problem, optimal Lyapunov function.

AMS Subject Classification: 34D20, 37C75, 49k15, 93D05, 93D30.

1. DIRECT CONTROL SYSTEMS

We will consider nonlinear Lur’e type control system [1], which is described in terms of nonlinear differential equations

\[ \dot{x}(t) = Ax(t) + bf(\sigma(x(t))), \sigma(x(t)) = c^T x(t) \]  

(1)

Here \( b, c, x(t) \in \mathbb{R}^n \), \( A \in \mathbb{R}^{n \times n} \) - square asymptotically stable matrix. Nonlinear scalar function \( f(\sigma) \) satisfies to the "sector conditions"

\[ 0 < f(\sigma) \sigma < k\sigma^2, k = \text{const} \]  

(2)

Asymptotic stability of zero solution \( x(t) \equiv 0 \) of the system (1) for an arbitrary \( f(\sigma) \), which satisfies conditions (2), was called the absolute stability [1].

As a rule, for an investigation of the system (1), the Lyapunov function looks as a sum of quadratic form from positive definite matrix \( H \) and integral of nonlinearity

\[ V(x) = x^T H x + \beta \int_0^{\sigma(x)} f(\sigma) d\sigma, \beta > 0. \]  

(3)

Denote

\[ C(H, \beta, \nu) = \begin{bmatrix} -A^T H - HA & -[Hb + (\beta A^T + I \nu c)]^T \\ -[Hb + (\beta A^T + I \nu c)] & \nu/k - \beta b^T c \end{bmatrix} \]

\( I \) - unique matrix.

The following conditions of absolute stability, which are obtained by using the Lyapunov function (3), take place.
Theorem 1. [1, 2] Let there exist the positive definite matrix $H$ and constants $\beta > 0, \nu > 0$ such that the matrix $C(H, \beta, \nu)$ is positive definite too. Then the direct control system (1) is absolute stable.

2. Optimization Method in the Finding of Absolute Stability Conditions of Direct Control Systems

Let us consider the problem of getting the guaranteed absolute stability conditions in given class of the Lyapunov functions, i.e. finding positive definite matrixes $H$ such that the minimal eigenvalue of symmetric matrix $C(H^0, \beta^0, \nu^0)$, which determines the derivative, will be maximal.

Optimization problem is considered on set of triples $L = (H, \beta, \nu) : H \geq 0, \beta \geq 0, \nu \geq 0$, where under $H \geq 0$ we mean that the matrix $H$ is the positive semidefinite.

As is known, a symmetric matrix $C(H, \beta, \nu)$ is positive definite if and only if

\[ \lambda_{min}[C(H, \beta, \nu)] > 0 \]

($\lambda_{min}(\cdot)$ is the minimal eigenvector of corresponding matrices). And the problem of finding the guaranteed absolute stability condition of the system (1) in class of functions (3) can be present as the optimization problem

\[ \phi_1 = (H, \beta, \nu) \rightarrow \min \lambda_{min}[C(H, \beta, \nu)] \quad \text{subject to} \quad H \geq 0, \beta \geq 0, \nu \geq 0 \]  

(4)

with restrictions

\[ \lambda_{min}(H) \geq 0, \beta \geq 0, \nu \geq 0, \phi_1(H, \beta, \nu) = -\lambda_{min}[C(H, \beta, \nu)]. \]  

(5)

It is easy to see that the set $L$ - is a linear space, which is a convex cone.

And, if the optimization problem (4), (5) has the solution $(H^0, \beta^0, \nu^0)$, for which $\phi_1(H^0, \beta^0, \nu^0) < 0$, then the direct control system (1) will be absolutely stable.

If $\phi_1(H^0, \beta^0, \nu^0) > 0$ then the problem of absolute stability investigation in class of functions (3) by selection of parameters $H, \beta, \nu$ can not be solved.

Denote by $L_1$ subset of $L$, which consists of triples $(H, \beta, \nu)$ belong to the unit sphere, i.e. satisfy condition

\[ \lambda_{max}^2(H) + \beta^2 + \nu^2 \leq 1 \]  

(6)

Lemma 1. The optimization problem (4)-(6) has a solution.

Lemma 2. The function $\phi_1(H^0, \beta^0, \nu^0)$ on the set $L_1$ is a convex function.

Accordingly, problem (4)-(6) is the problem of convex optimization. Extremal eigenvalues of symmetric positive definite matrixes are piecewise continuously differentiable functions. Therefore, the solution existence conditions will be formulated in terms of a sub-gradient [2].

Definition 1. The scalar product of two triples $(H_1, \beta_1, \nu_1)$ and $(H_2, \beta_2, \nu_2)$ is

\[ \langle (H_1, \beta_1, \nu_1), (H_2, \beta_2, \nu_2) \rangle = \sum_{i,j=1}^{n} h_{ij} h_{ij}^2 + \beta_1 \beta_2 + \nu_1 \nu_2, \]

where $H_1 = \{h_{ij}^1\}, H_2 = \{h_{ij}^2\}, i, j = 1, n$.

Denote as $\Delta_{ij}$ - symmetric $(n \times n)$ matrix, which have as $(i, j)$ and $(j, i)$ elements the constant 1/2 in case $i \neq j$ and the 1, in case $i = j$. $\Theta$ - is the zero matrix. Then arbitrary symmetric matrices $H_1 = \{h_{ij}^1\}, H_2 = \{h_{ij}^2\}, h_{ij} = 1, n$ can be represented as the decomposition

\[ H_1 = \sum_{i,j=1}^{n} h_{ij} \Delta_{ij}, \quad H_2 = \sum_{i,j=1}^{n} h_{ij} \Delta_{ij} \]
**Definition 2.** The sub-gradient of convex function \( \phi_1(H, \beta, \nu) \) at internal point \((H^0, \beta^0, \nu^0) \in L_1 \) denote triple \((E_0, f_0, k_0)\) for which, for any \((H, \beta, \nu) \in L_1 \) will be fulfilled
\[
\phi_1(H, \beta, \nu) - \phi_1(H^0, \beta^0, \nu^0) \geq \langle (E_0, f_0, k_0), (H - H^0, \beta - \beta^0, \nu - \nu^0) \rangle.
\]

**Theorem 2.** The sub-gradient of the function
\[
\phi_1(H, \beta, \nu) = -\lambda_{\min}[C(H, \beta, \nu)]
\]
at internal point \((H^0, \beta^0, \nu^0) \in L_1 \) is triple \((E_0, f_0, k_0)\) which consists of \(E_0\), scalars \(f_0, k_0\) and has next form
\[
E_0 = \{ e_{ij} \}, \quad e_{ij}^0 = -z_0^T C(\Delta_{ij}, 0, 0) z_0, \quad i, j = 1, n,
\]
\[
f_0^0 = -z_0^T C(\Theta, 1, 0) z_0,
\]
\[
k_0^0 = -z_0^T C(\Theta, 0, 1) z_0.
\]

Here \(z_0\) is unit vector on which quadratic form \(z^T C(H^0, \beta^0, \nu^0)z\) reaches minimal value (eigen-vector corresponding to minimal eigenvalue).

Using the obtained expression for sub-gradient and fact of convex set \(L_1\) we will formulate solvability conditions of optimization problem (4) - (6), accordingly to [3].

**Theorem 3.** The function \( \phi_1(H, \beta, \nu) \) achieve their minimal value at the point \((H^0, \beta^0, \nu^0) \in L_1 \) if and only if, when for arbitrary \((H, \beta, \nu) \in L_1\) will be true next condition
\[
\langle (E_0, f_0, k_0), (H - H^0, \beta - \beta^0, \nu - \nu^0) \rangle \geq 0.
\]

Moreover point \((H^0, \beta^0, \nu^0)\) satisfied boundary conditions
\[
\sum_{i,j=1}^{n} (h_{ij}^0)^2 + (\beta^0)^2 + (\nu^0)^2 = 1.
\]

Thus, absolute stability conditions of system (1) can be formulated as follows.

**Theorem 4.** Let is \(H^0\) - positive definite matrix and scalar parameters \(\beta^0, \nu^0\) such that
\[
\langle (E_0, f_0, k_0), (H - H^0, \beta - \beta^0, \nu - \nu^0) \rangle \geq 0, \lambda_{\max}^2(H^0) + (\beta^0)^2 + (\nu^0)^2 = 1.
\]

Then for absolute stability of the system (1) sufficient that matrix \(C(H^0, \beta^0, \nu^0)\) will be positive definite.

Moreover, if \(C(H^0, \beta^0, \nu^0)\) is not positive definite, then we can not obtain any theorem about absolute stability for system (1) with utilization of Lyapunov function from apriority given class of function (3). Namely, function
\[
V_0(x) = x^T(t)H^0x(t) + \beta^0 \int_{0}^{\sigma(x)} f(\sigma)d\sigma.
\]
is an “optimal” in given class.

**Acknowledgments**

The first author was supported by the Grant of research project MeMoV EV90800005/21400 c.h. CZ.02.2.69/0.0/0.0/16_027/0008371.01/06/2018.

**References**

VARIATION FORMULAS FOR DELAY DIFFERENTIAL EQUATIONS AND NECESSARY OPTIMALITY CONDITIONS

TEA SHAVADZE

1Ivane Javakhishvili Tbilisi State University, Department of Mathematics & I. Vekua Institute of Applied Mathematics, 2 University Str., Tbilisi 0186, Georgia

e-mail: tea.shavadze@gmail.com

Let \( O \subset \mathbb{R}^n \) and \( U_0 \subset \mathbb{R}^r \) be open sets. Let \( \theta_{i2} > \theta_{i1} > 0, i = 1, s \) be given numbers and \( n \)-dimensional function \( f(t, x, x_1, ..., x_s, u) \) satisfies the following conditions: for almost all fixed \( t \in I = [a, b] \) the function \( f(t, \cdot, \cdot) : O^{1+s} \times U_0 \to \mathbb{R}^n \) is continuously differentiable; for each fixed \( (x, x_1, ..., x_s, u) \in O^{1+s} \times U_0 \) the functions \( f(t, x, x_1, ..., x_s, u), f_x(t, \cdot), f_s(t, \cdot), i = 1, s \) and \( f_u(t, \cdot) \) are measurable on \( I \); for compact sets \( K \subset O \) and \( U \subset U_0 \) there exists a function \( m_{K, U}(t) \in L_1(I, [0, \infty)) \) such that

\[
| f(t, x, x_1, ..., x_s, u) | + | f_x(t, \cdot) | + \sum_{i=1}^{s} | f_s(t, \cdot) | + | f_u(t, \cdot) | \leq m_{K, U}(t),
\]

for all \( (x, x_1, ..., x_s, u) \in K^{1+s} \times U \) and for almost all \( t \in I \). Furthermore, \( \Phi \) is the set of continuous initial functions \( \varphi : I_1 = [\hat{\tau}, b] \to O, \hat{\tau} = a - \max \{ \theta_{i1}, ..., \theta_{is} \} \) and let \( \Omega \) be a set of measurable functions \( u(t), t \in I \) satisfying the condition \( du(I) \subset U_0 \) and it is compact in \( \mathbb{R}^r \).

To each element \( \mu = (t_0, \tau_1, ..., \tau_s, x_0, \varphi, u) \in \Lambda = [a, b] \times [\theta_{i1}, \theta_{i2}] \times ... \times [\theta_{s1}, \theta_{s2}] \times O \times \Phi \times \Omega \) we assign the delay controlled differential equation

\[
\dot{x}(t) = f(t, x(t), x(t - \tau_1), ..., x(t - \tau_s), u(t)), \quad (1)
\]

with the discontinuous initial condition

\[
x(t) = \varphi(t), \quad t < t_0, \quad x(t_0) = x_0. \quad (2)
\]

The condition (2) is said to be the discontinuous initial condition since, in general, \( x(t_0) \neq \varphi(t_0) \).

**Definition 1.** Let \( \mu = (t_0, \tau_1, ..., \tau_s, x_0, \varphi, u) \in \Lambda \). A function \( x(t) = x(t; \mu) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b) \) is called a solution of equation (1) with the initial condition (2) or the solution corresponding to \( \mu \) and defined on the interval \( [\hat{\tau}, t_1] \) if it satisfies condition (2) and is absolutely continuous on the interval \([t_0, t_1]\) and satisfies equation (1) almost everywhere (a.e.) on \([t_0, t_1]\).

Let \( \mu_0 = (t_{00}, \tau_{10}, ..., \tau_{s0}, x_{00}, \varphi_0, u_0) \in \Lambda \) be a fixed element and let \( x_0(t) \) be the solution corresponding to \( \mu_0 \) and defined on the interval \([\hat{\tau}, t_{10}]\), where \( t_{00}, t_{10} \in (a, b), t_{00} < t_{10} \) and \( \tau_{i0} \in (\theta_{i1}, \theta_{i2}), i = 1, s \).

Let us introduce the set of variation:

\[
V = \{ \delta \mu = (\delta t_0, \delta \tau_1, ..., \delta \tau_s, \delta x_0, \delta \varphi, \delta u) : \delta t_0 \in (a, b) - t_{00}, \delta \tau_i \in (\theta_{i1}, \theta_{i2}) - \tau_{i0}, i = 1, s, \}
\]

\[
\delta x_0 \in O - x_{00}, \delta \varphi = \sum_{i=1}^{s} \lambda_i \delta \varphi_i, \delta \varphi_i \in \Phi - \varphi_0, \delta u \in \Omega - u_0, | \delta t_0 | \leq \alpha, | \delta \tau_i | \leq \alpha, i = 1, s, \]

\[
| \delta x_0 | \leq \alpha, | \lambda_i | \leq \alpha, i = 1, k, || \delta u || \leq \alpha, \}
\]
where \( \alpha > 0 \) is a fixed number, \( (a, b) - t_{00} = \{ \delta t_{0} = t_{0} - t_{00} : \forall t_{0} \in (a, b) \} \) and \( ||\delta u|| = sup \{ ||\delta u(t)|| : t \in I \} \).

There exist numbers \( \delta_{1} > 0 \) and \( \epsilon_{1} > 0 \) such that for arbitrary \( (\epsilon, \delta \mu) \in (0, \epsilon_{1}) \times V \) we have \( \mu_{0} + \epsilon \delta \mu \in \Lambda \), and the solution \( x(t; \mu_{0} + \epsilon \delta \mu) \) defined on the interval \( [\tilde{t}, t_{10} + \delta_{1}] \subset I_{1} \) corresponds to it \([1]\). By the uniqueness, the solution \( x(t; \mu_{0}) \) is a continuation of the solution \( x_{0}(t) \) on the interval \( [\tilde{t}, t_{10} + \delta_{1}] \). Therefore, we can assume that the solution \( x_{0}(t) \) is defined on the whole interval \( [\tilde{t}, t_{10} + \delta_{1}] \).

Now we introduce the increment of the solution \( x_{0}(t) := x(t; \mu_{0}) \),

\[
\Delta x(t; \epsilon \delta \mu) = x(t; \mu_{0} + \epsilon \delta \mu) - x_{0}(t), \forall (t, \epsilon, \delta \mu) \in [\tilde{t}, t_{10} + \delta_{1}] \times (0, \epsilon_{1}) \times V. \tag{3}
\]

**Theorem 1.** Let the following conditions hold:

1.1) \( \tau_{s0} > \cdots > \tau_{10} \) and \( t_{00} + \tau_{s0} < t_{10} \);

1.2) the function \( \varphi_{0}(t) \) is absolutely continuous and \( \varphi_{0}(t), t \in I_{1} \) is bounded;

1.3) the function \( f(w, u) \), where \( w = (t, x, x_{1}, \ldots, x_{s}) \in I \times O^{1+s} \) is bounded on \( I \times O^{1+s} \times U_{0} \);

1.4) there exists the finite limit

\[
\lim_{w_{0} \to w} f(w, u_{0}(t)) = f^{-}, \text{ } w_{0} \in (a, t_{00}) \times O^{1+s},
\]

where \( u_{0} = (t_{00}, x_{00}, \varphi_{0}(t_{00} - \tau_{10}), \ldots, \varphi_{0}(t_{00} - \tau_{s0})) \);

1.5) there exist the finite limits

\[
\lim_{(w_{11}, w_{21}) \to (w_{1i}, w_{2i})} \left[ f(w_{11}, u_{0}(t)) - f(w_{21}, u_{0}(t)) \right] = f_{i},
\]

where \( w_{1i}, w_{2i} \in (a, b) \times O^{1+s}, i = 1, k, \)

\[
w_{1i} = (t_{00} + \tau_{0i}, x_{0}(t_{00} + \tau_{i0})), x_{0}(t_{00} + \tau_{i0} - \tau_{10}), \ldots, x_{0}(t_{00} + \tau_{i0} - \tau_{10}),
\]

\[
x_{0}(t_{00}, x_{0}(t_{00} + \tau_{i0} - \tau_{i10}), \ldots, x_{0}(t_{00} + \tau_{i0} - \tau_{i10})),
\]

\[
w_{2i} = (t_{00} + \tau_{0i}, x_{0}(t_{00} + \tau_{i0})), x_{0}(t_{00} + \tau_{i0} - \tau_{10}), \ldots, x_{0}(t_{00} + \tau_{i0} - \tau_{10}),
\]

Then there exist numbers \( \epsilon_{2} \in (0, \epsilon_{1}) \) and \( \delta_{2} \in (0, \delta_{1}) \) with \( \tau_{10} - \delta_{2} > t_{00} + \tau_{s0} \) such that for arbitrary \( (t, \epsilon, \delta \mu) \in [t_{10} - \delta_{2}, t_{10} + \delta_{2}] \times (0, \epsilon_{2}) \times V^{-} \), where \( V^{-} = \{ \delta \mu \in V : \delta t_{0} \leq 0 \} \), we have

\[
\Delta x(t; \epsilon \delta \mu) = \epsilon \delta x(t; \delta \mu) + o(t; \epsilon \delta \mu). \tag{4}
\]

Here

\[
\delta x(t; \delta \mu) = -Y(t_{00}; t)f^{-} \delta t_{0} + \beta(t; \delta \mu),
\]

\[
\beta(t; \delta \mu) = Y(t_{00}; t)\delta x_{0} - \left[ \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t)f_{i} \right] \delta t_{0} - \left[ \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t)f_{i} \right] \delta \tau_{i}
\]

\[
+ \int_{t_{00}}^{t} Y(\xi; t)f_{x}[\xi] \delta \varphi(\xi - \tau_{00})d\xi
\]

\[
+ \sum_{i=1}^{s} \int_{t_{00} - \tau_{i0}}^{t} Y(\xi + \tau_{i0}; t)f_{x}[\xi + \tau_{i0}] \delta \varphi(\xi)d\xi + \int_{t_{00}}^{t} Y(\xi; t)f_{u}[\xi] \delta u(\xi)d\xi, \tag{5}
\]

where is assumed that

\[
\int_{t_{00}}^{t} Y(\xi; t)f_{x}[\xi] \delta \varphi(\xi - \tau_{00})d\xi = \int_{t_{00} + \tau_{00}}^{t} Y(\xi; t)f_{x}[\xi] \delta \varphi(\xi - \tau_{00})d\xi + \int_{t_{00} + \tau_{00}}^{t} Y(\xi; t)f_{x}[\xi] \delta \varphi(\xi - \tau_{00})d\xi.
\]
Next, \( Y(\xi; t) \) is the \( n \times n \)-matrix function satisfying the equation

\[
Y_\xi(\xi; t) = -Y(\xi; t)f_\varepsilon[\xi] - \sum_{i=1}^{s} Y(\xi + \tau_0; t)f_{\varepsilon, i}[\xi + \tau_0], \xi \in [t_0, t]
\]

and the condition

\[
Y(\xi; t) = \begin{cases} H \text{ for } \xi = t, \\ \Theta \text{ for } \xi > t; \end{cases}
\]

\( H \) is the identity matrix and \( \Theta \) is the zero matrix; \( f_{\varepsilon, i} = \frac{\partial}{\partial \varepsilon} f_{x_i}[\xi] = f_{x_i}(\xi, x_0(\xi), x_0(\xi - \tau_0), \ldots, x_0(\xi - \tau_{s0}), u_0(\xi)) \); \( \lim_{\varepsilon \to 0} o(t; \varepsilon \delta \mu) / \varepsilon = 0 \) uniformly for \((t, \delta \mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times V^-\).

The function \( \delta x(t; \delta \mu) \) is called the first variation of the solution \( x_0(t) \). The expression (5) is called the variation formula of solution.

**Theorem 2.** Let the conditions 1.1)-1.3) and 1.5) of the Theorem 1 hold. Moreover, there exists the finite limit

\[
\lim_{w \to u_0} f(w, u_0(t)) = f^+, w \in [t_0, b] \times O^{1+s}.
\]

Then there exist numbers \( \varepsilon_2 \in (0, \varepsilon_1) \) and \( \delta_2 \in (0, \delta_1) \), with \( t_{10} - \delta_2 > t_0 + \tau_{s0} \) such that for arbitrary \((t, \varepsilon, \delta \mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V^+\), where \( V^+ = \{ \delta \mu \in V : \delta t_0 \geq 0 \} \), the formula (4) holds, where

\[
\delta x(t; \delta \mu) = -Y(t_00; t)f^+ \delta t_0 + \beta(t; \delta \mu).
\]

The Theorems 2 and 3 are proved by a scheme given in [1].

**Theorem 3.** Let the conditions 1.1)-1.5) of the Theorem 1 and the condition (6) hold. Moreover, \( f^- = f^+ := \hat{f} \). Then there exist numbers \( \varepsilon_2 \in (0, \varepsilon_1) \) and \( \delta_2 \in (0, \delta_1) \), with \( t_{10} - \delta_2 > t_0 + \tau_0 \) such that for arbitrary \((t, \varepsilon, \delta \mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V\), the formula (4) holds, where

\[
\delta x(t; \delta \mu) = -Y(t_00; t)f \delta t_0 + \beta(t; \delta \mu).
\]

Theorem 3 is a corollary to Theorems 1 and 2.

Finally, we note that on the basis of variation formulas the necessary optimality conditions are obtained for the optimization problems with several delays, general boundary conditions and functional.

**Acknowledgments**

This work is supported by the Shota Rustaveli National Science Foundation, Grant No. PhD-F-17-89, Project Title: "Variation formulas of solutions for controlled functional differential equations with the discontinuous initial condition and considering perturbations of delays and their applications in optimization problems ".

**Keywords:** Variation formula, delay differential equation, optimization.

**AMS Subject Classification:** 34K27, 34K99, 49J21, 49K40.

**References**

FUZZY APPROACH TO DETERMINE THE THICKNESS OF THE OIL SLICK ON THE WATER SURFACE

R.Y. SHIKHLINSKAYA1, R.Y. AHMADOVA2, F.A. MIRZAYEV3

1Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
2Geography Institute of ANAS, Baku, Azerbaijan
3Baku State University, Department of Economic Cybernetics

e-mail: reyhanshikhli@gmail.com, rena.ahmedova.67@mail.ru, farhad_1958@mail.ru

1. Introduction

To accurately and effectively eliminate the accident at sea, minimize its negative impact, models that accurately determine the characteristics of the oil spill occurred during the accident should be developed.

The development of modern information technology allows for a more adequate solution to the issue. Fuzzy mathematical apparatus, where indicators are fuzzy, translates mathematical and logical operations into algorithms and uses them in computer programs [1].

Let’s first define the fuzzy set [2, 6].

Definition. A fuzzy set \( A \) in \( X \) is characterized by a membership function \( f_A(x) \) which associates with each point in \( X \) a real number in the interval \( [0,1] \), with the values of \( f_A(x) \) at \( x \) representing the “grade of membership” of \( x \) in \( A \). Thus, the nearer the value of \( f_A(x) \) to unity, the higher the grade of membership of \( x \) in \( A \). As seen from the definition the universal multiplicity can be mass of people, products, properties, indicators, business and so on. The affiliation function of each element taken from this cluster varies from 0 to 1 and indicates to which degree the corresponding element belongs to the multitude.

Obviously, the element is in the multitude with the value \( \mu_A(x) = 1 \), and with the value \( \mu_A(x) = 0 \), the element is not in the multitude. \( \mu_A(x) \) increases as increasing the degree of belonging to the element.

2. Formulation of the problem

The aim of the study is to build a fuzzy model that reflects the concentration of oil and the effects of water temperature on the thickness of the oil spill. An intelligent system based on fuzzy theory and implemented in the FUZZY LOGIC subsystem of MATLAB TOOLBOX program package was set up to determine this dependency [4].

The fuzzy logic scheme consists of the following steps for solving the problem [3]:

(1) Determining key input and output indicators affecting the process. Setting of term-clusters and its phazification.
(2) Collecting and formulating knowledge about the mutual effect of the indicators. Setting knowledge base.
(3) Processing of indicators based on fuzzy logical outcomes.

[7] once the knowledge was obtained from the experts, the following fuzzy model is setting up to determine the radius of the oil spread:
After the necessary knowledge is extracted from experts, set the following fuzzy model to determine the thickness of the oil slick

\[ D = (K, t) \]  

(1)

Where \( K \) and \( t \) are input, \( D \) is output variables:
- \( D \) - thickness of oil slick [0 – 8] (mkm)
- \( K \) - oil concentration, [0 – 4] (mg/l)
- \( t \) – water temperature [3 – 28]

The model will determine the thickness of the oil spill according to the specific values of the input variables.

3. Solution of the problem

1. Determining key input and output indicators affecting the process. Determination of fuzzy linguistic terms and its fuzzification.

For each variable, we define three fuzzy linguistic term sets as follows:
- \( D \) = \{very small, small, average, large, very large\}
- \( K \) = \{very low, low, average, high, very high\}
- \( t \) = \{very low, low, average, high, very high\}

It is possible to set input and output variables using the FUZZY LOGIC subsystem of MATLAB TOOLBOX program package. For example, the ”oil concentration” variable is described as follows:

![Figure 1. Description of the ”oil concentration” fuzzy variable.](image.png)

2. Collecting and formulating of knowledge about the effect of the indicators. Establishing of knowledge base.

Once the linguistic indicators and their interval (carriers) are entered into the model, a knowledge base is formed on the basis of the logical implication scheme ‘IF-THEN’’. The knowledge base consists of rules. Each rule allows any value of the input variables on the left side to set the value for the output variable on the right.

\[ IF x_1 = \tilde{A}_1 \land x_2 = \tilde{A}_2 \land x_3 = \tilde{A}_3 \land x_4 = \tilde{A}_4 \land x_5 = \tilde{A}_5 \land x_6 = \tilde{A}_6 \land x_7 = \tilde{A}_7 THEN y = \tilde{B}, \]

(2)

where \( x_i = A_i, i = 1,7 \) fuzzy terms corresponding to indicator.

The proposed model uses six rules.

3. Processing of indicators based on fuzzy logical outcomes in the FUZZY LOGIC subsystem of MATLAB TOOLBOX program package. At this stage, the thickness of the oil spill is determined by the specific value of oil concentration and water temperature:
Figure 2. Fuzzy Logical Outsourcing Scheme, Determining the Oil Spray Concentration: If $K = 2\text{mkmandt} = 15^\circ$ then $D = 4.01$.

Figure 3. Dependence between water temperature and concentration of oil.

**Keywords:** Fuzzy logic, fuzzy sets, fuzzy inference systems, oil slick.

**AMS Subject Classification:** 93C42.

**REFERENCES**


LOCAL MESHLESS METHOD FOR CONVECTION DOMINATED STEADY AND UNSTEADY PARTIAL DIFFERENTIAL EQUATIONS

SIRAJ-UL-ISLAM

1Department of Basic Sciences, University of Engineering and Technology, Peshawar, Pakistan
e-mail: siraj.islam@gmail.com

ABSTRACT. In this paper, we propose a mildly shock capturing stabilized local meshless method (SLMM) for convection dominated steady and unsteady PDEs. This work is extension of the numerical procedure, which was designed only for steady state convection dominated PDEs [2]. The proposed meshless methodology is based on employing different types of stencils embodying the already known flow direction. Numerical experiments are performed to compare the proposed method with the finite-difference method on special grid (FDSG) and other numerical methods. Numerical results confirm that the new approach is accurate and efficient for solving a wide class of one- and two-dimensional convection dominated PDEs having sharp corners and jump discontinuities. Performance of the SLMM is found better and comparable than the mesh-based numerical methods.

Keywords: Local Meshless collocation method, radial basis function, boundary layer, convection-diffusion-PDEs, TVD Runge-Kutta Method.

AMS Subject Classification: 65Mxx, 65Nxx, 65Yxx, 65Zxx.

1. INTRODUCTION

Numerical solution of steady and unsteady convection dominated diffusion reaction problems pose major challenges to the available numerical methods. These equations appear in many engineering problems such as fluid dynamics modeling, electrochemical interactions, chemically reactive flows, weather prediction, ocean circulation, reservoir simulation etc. The common source of computational difficulties and wiggly solution is related to advectivtion phenomena, when the Peclet number reaches higher values. In such cases, the solution is contaminated by spurious oscillation, numerical smearing, peak clipping, while capturing steep fronts and shocks. Some well known mesh-based and meshless methods appeared in the literature recently, have partially addressed these problems (see [1, 3, 4] and the references therein) in the context of numerical solution of such PDEs.

In this work we use a local collocation meshless procedure for capturing mild jumps and shocks, appearing in the steady state and time dependent PDEs. The beauty of this procedure is handling of shocks and steep fronts by manipulation of the local interpolation domain called stencil. Stencil formation embodying flow dynamical features is the back bone of the proposed local meshless method, which can be implemented on both uniform and non-uniform nodes. Meshless methods are free from mesh connectivity and do not need meshing and re-meshing as required in the case of finite-element method. For the time stepping, TVD Runge-Kutta
methods of order 2 and 3 are used. Such type of time integration schemes ensure that the total variation of the solution does not increase with the advancement in time.

2. Numerical experiments

Test Problem 1. Consider a linear one-dimensional advection dominated equation representing mathematical modeling of batch crystallization process [5]. Initial data for the problem is given as follows:

\[
u_0(x, 0) = \begin{cases} 
0.0 & \text{if } x \leq 2\mu m, \\
10^{10} & \text{if } 2\mu m \leq x \leq 10\mu m, \\
0.0 & \text{if } 10\mu m \leq x \leq 18\mu m, \\
10^{10}\cos^2\left(\frac{\pi(x-26)}{64}\right) & \text{if } 18\mu m \leq x \leq 34\mu m, \\
0.0 & \text{if } 34\mu m \leq x \leq 42\mu m, \\
10^{10}\sqrt{1 - \frac{(x-50)^2}{64}} & \text{if } 42\mu m \leq x \leq 58\mu m, \\
0.0 & \text{if } 58\mu m \leq x \leq 66\mu m, \\
10^{10}e^{-\frac{x(x-70)}{2x^2}} & \text{if } 66\mu m \leq x \leq 74\mu m, \\
0.0 & \text{if } x \geq 74\mu m.
\]

Analytical solution corresponding to the initial data \(u_0\) is given as:

\[u(x, t) = u_0(x - Gt).\]

In the above equation, \(G\) is a constant crystal growth rate. In our case, we consider \(G = 0.1\mu m/sec, \sigma = 0.668\Delta x\) and final time \(t = 60\) sec. We use the TVD Runge-Kutta method of order 2 for time integration. As apparent from the crystal size density (CSD), the solution contains different variations in the form of sharp edges and peaks of magnitude \(10^{10}\) [5], thus making it a hard problem for the available numerical methods. We test performance of the proposed SLMM to capture accurate CSD, which is free from unrealistic oscillatory behavior. Several methods[5, 3] have been reported in the literature for finding the numerical solution of this problem. The SLMM is implemented for the batch crystallization process accurately. We have compared numerical solution of the proposed method with available methods in[5]. Numerical solution of the SLMM is better than LBRFCM[5] and other methods implemented in [3] for solving the same batch crystallization problem.

Test Problem 2. Consider one-dimensional nonlinear Burger’s equation with Dirichlet type boundary conditions having analytical solution[5]

\[u(x, t) = \frac{0.1P + 0.5Q + R}{P + Q + R}, \quad (x, t) \in (0, 2) \times (0, 2),\]

where

\[P = e^{-0.05(x-0.5+4.95)}, \quad Q = e^{-0.25(x-0.5+0.75)}, \quad R = e^{-0.05(x-0.375)}\]

The solution initially has two steep fronts. Steepness of the two fronts reduces with the time and eventually merge into a single front. The steepness and merging of front make this problem hard to capture accurate solution with advancement of time. In [5], numerical results obtained by the LBRFCM are presented for the same problem. It is reported that the numerical solution did not catch up the exact solution well for large values of time. We use the TVD RK method of order 2 for time integration.
3. Conclusion

A local meshless collocation method is used for numerical solution of CDR type of steady and unsteady PDEs. A stable solution is obtained by the virtue of different stencil based upwind strategies designed for such PDEs. Accurate solution has been obtained in the presence of shock, moving cylinder and internal boundary layer phenomena. The merits of the current work include, intelligent use of flow dynamic driven stencil, embodying automatic stabilization procedure. The demerits of the current approach include the requirement of priori flow direction information and finding an optimized value of the shape parameter.

References

A NEW DEEP LEARNING BASED SYSTEM FOR FACIAL GESTURE ANALYSIS

BÜŞRA EMER SOYLU, MEHMET SERDAR GÜZEL, İ. N. ASKERZADE

1. Introduction

Facial gestures carry critical information as non-verbal communication and are considered one of the critical problem of computer vision. Deep learning is emerging as a powerful approach for machine learning and has achieved satisfactory performance in many areas. This paper proposes a novel facial gestures analysis system, implementing a novel deep neural network structure based on a Convolutional Neural Network (CNN) architecture. Seven different gesture classes are defined for the facial gesture analysis. Various facial images are obtained from different comprehensive datasets to train and verify the overall performance of the proposed system. Having observed the performance of the proposed system on the facial gesture analysis problem, it is first compared with a conventional system currently based on objective evaluation parameters, which is then compared with recent studies that rely on deep learning for the same problem. Experimental results, based on JAFFE, KDEF, and MUG databases, verify the superiority of the proposed system over the conventional approach and also deep learning based systems.

2. Basic Concepts

Facial gestures are the most determinative, powerful, natural, and nonverbal universal signs to understand a person’s psychological state during communication. It is expected that the computers will also gain the ability to recognize facial gestures in order to interact effectively with people. The aim of facial gestures analysis (FGA) is to analyze and recognize facial movements automatically, and also to estimate changes in facial features using visual information [4].

This paper first presents a popular conventional system used at present. Afterwards, a novel deep learning system, implementing a CNN based architecture, is introduced. The analysis of seven gesture states (happiness, sadness, anger, fear, disgust, surprised, and neutral) are performed by utilizing these systems based on objective evaluation parameters. Besides, the proposed system is also compared with recent studies using deep learning for the FGA problem.

2.1. Histogram of Oriented Gradients. The conventional FGA system employs a Histogram of Oriented Gradients (HOG) for feature extraction and Support Vector Machine (SVM) for classification problem respectively. Essentially, this conventional system first needs a reliable feature extraction methodology. Facial features are the prominent attributes of eyebrows, eyes, nose, and mouth. Gestures are formed by the deformation of these features, and the temporary appearance of the face. The most critical issue for a successful facial gesture recognition system is to design a robust and efficient face feature descriptor and to extract facial features successfully.
Kumari et al. (2015) stated the most popular techniques used to extract facial features, which is namely the HOG and detailed in the following sub-section [3].

2.1.1. Histogram of Oriented Gradients. HOG is a gradient-based method that consists of histograms which are calculated according to density directions in an image. HOG was first proposed by Dalal and Triggs [2]. This method, originally developed for object detection, is used in a wide range of applications. This feature descriptor algorithm consists of several steps. First, gradients and orientation histograms are calculated.

The horizontal gradient $G_x$ and vertical gradient $G_y$ of the images are calculated with the given equations.

$$G_x(x, y) = I(x + 1, y) - I(x - 1, y), \quad (1)$$

$$G_y(x, y) = I(x, y + 1) - I(x, y - 1). \quad (2)$$

By using these gradient components, the gradient size $|G|$ of the image and the gradient orientations $\phi$ are computed using equations 5 and 6.

$$|G| = \sqrt{G_x(x, y)^2 + G_y(x, y)^2}, \quad (3)$$

$$\phi = \tan^{-1}\frac{G_y(x, y)}{G_x(x, y)}. \quad (4)$$

The given problem is not a two-class problem because there are seven separate classes, as presented. However, for this problem, two-class classifiers are implemented by employing one-against-all method in which there exists one binary classifier for each class to separate members of that class from members of other classes.

![Figure 1. Multi-class classifier flow diagram.](image)

2.2. Conventional Neural Network for FGA Problem. This article presents a novel CNN based system for the FGA problem. The proposed system architecture is illustrated in Figure 2. The architecture and corresponding hyper parameters are generated after a series of preliminary trials and tests. The network consists of three convolution layers which are mainly responsible to extract distinctive features. The parameters of this layer consist of learnable filters which can be run as feature detectors. Higher-level convolutional layers are expected to extract higher-level features, whereas the first convolutional layer extracts simple edges, lines, corners, and other low-level features.

2.3. Performance Evaluation. As expected, the number of data, camera axis and illumination, quality, and resolution difference in the datasets affects the success rate. Besides, due to the challenging characteristics of the facial gesture analysis problem and limited data constraint, the conventional system has not achieved the desired success, especially in Jaffe and KDEF datasets. However, the proposed model achieved a reasonable recognition accuracy which reached 91.18% for Jaffe and 94.49% MUG datasets. The only dataset where the conventional approach exceeds an 80% success rate is the MUG, having more data samples than other datasets. Nevertheless, the proposed approaches reach almost a 95% success rate for this dataset.
3. Conclusions

This paper proposes a new deep learning based system for facial gesture analysis (FGA). The system is designed based on a specific CNN architecture for the FGA problem. A series of experiments were carried out to estimate the most appropriate hyper parameters for the system. Performance evaluations were performed using three comprehensive image datasets, namely JAFFE, KDEF, and MUG. Experimental results prove the superiority of the proposed architecture over the conventional method. Besides, the proposed deep learning based architecture is compared with the recent and popular FGA systems using deep learning methodologies. While cross-datasets are employed by different articles for performance evaluation, a performance comparison within the state-of-the-art systems utilized JAFFE and KDEF datasets. Results validate overall performance on benchmark datasets so that the proposed system achieves the highest precision rate with JAFFE image datasets having a limited number of training images. Besides, it can achieve comparable precision rates for KDEF datasets. Finally, this study also presents experimental results for the MUG dataset, which is not preferred by the state-of-the-art studies working on FGA. The proposed system also archives a high precision rate for the recognition of facial gestures for this dataset. The acceptance of a deep neural network has increased owing to its influential performance in very large data sets and its low cost. However, this is not the case for the FGA problem, where the number of data are limited. In future studies, a larger and more natural data set will be prepared for more advanced experiments. The authors also believe that the proposed model can easily be integrated into more advanced systems, aiming to perform FGA for 3D images and a real-time robotic system.

Keywords: Facial gesture analysis, deep learning, CNN, HOG, SVM.

AMS Subject Classification: 65D18, 62M40

References

OPTICAL SOLITONS TO THE CONFORMABLE TIME-FRACTIONAL PERTURBED RATHAKRISHNAN-KUNDU-LAKSHMANAN EQUATION

TUKUR ABDULKADIR SULAIMAN\textsuperscript{1,2}, HASAN BULUT\textsuperscript{1,3}, GULNUR YEL\textsuperscript{3}, SIBEL SEHRIBAN ATAS\textsuperscript{1}

\textsuperscript{1}Department of Mathematics, Firat University, Elazig, Turkey
\textsuperscript{2}Department of Mathematics, Federal University, Dutse, Jigawa, Nigeria
\textsuperscript{3}Department of Mathematics Education, Final International University, Kyrenia, Cyprus

e-mail: sulaiman.tukur@fud.edu.ng, hbulut@firat.edu.tr, gulnuryel33@gmail.com, sibel.s.atas@gmail.com

Abstract. This study reaches the dark and singular optical solitons solutions to the time-fractional Radhakrishnan-Kundu-Lakshmanan equation. The parametric conditions that guarantee the existence of valid solitons and other solutions are stated. The reported solutions may be useful in explaining the physical meaning of the Radhakrishnan-Kundu-Lakshmanan equation and other related nonlinear models arising in nonlinear sciences.

Keywords: Sinh-Gordon equation, optical soliton, fractional derivative.

AMS Subject Classification: 35C08, 47J35, 35R11.

1. Introduction

Several complex nonlinear physical phenomena can be expressed in form of nonlinear evolution equations (NLEEs). nonlinear Schrödinger’s type equations (NLSEs) are special type of nonlinear evolution equations that can be used to describe nonlinear physical aspects such as plasma physics, fluid dynamics, photonics, quantum electronics, and electromagnetism [1]. The theory of optical solitons is one of the interesting topics for the investigation of soliton propagation through nonlinear optical fibers. For over two decades, this area has attracted the attentions of many researchers.

However, this study is aimed at investigating the conformable time-fractional perturbed Radhakrishnan-Kundu-Lakshmanan equation [2] by ustilizing the extended sinh-Gordon equation expansion method (ShGEEM) [3, 4].

The dimensionless form of the conformable time-fractional perturbed Radhakrishnan-Kundu-Lakshmanan equation under the Kerr law nonlinearity is given as

\[ iD_t^\alpha \phi + a\phi_{xx} + b|\phi|^2\phi - i\delta \phi_x - i\lambda (|\phi|^2\phi)_x - i\sigma (|\phi|^2)_x\phi - i\gamma \phi_{xxx} = 0, \quad 0 < \alpha \leq 1. \]  

(1)
2. Applications

In this section, we employ the extended ShGEEM to the conformable time-fractional perturbed Radhakrishnan-Kundu-Lakshmanan. Consider the complex conformable time-fractional travelling wave transformation

\[ \phi(x, t) = \Psi(\zeta) e^{i\Theta}, \quad \zeta = \mu \left( x - c \frac{t^\alpha}{\alpha} \right), \quad \Theta = -kx + \omega \frac{t^\alpha}{\alpha} + \theta. \tag{2} \]

Substituting Eq. (2) into Eq. (1), gives

\[ \mu^2 (a + 3k\gamma) \Psi'' + (b - k\lambda) \Psi^3 - (\omega + ak^2 + \delta k + \gamma k^3) \Psi = 0 \tag{3} \]

from the real part, and

\[ \mu^2 \gamma \Psi'' - (c + 2ak + \delta + 3k^2\gamma) \Psi' - (3\lambda + 2\sigma) \Psi^2 \Psi' = 0 \tag{4} \]

from the imaginary part.

Integrating Eq. (4) once, we have

\[ 3\mu^2 \gamma \Psi'' - 3(c + 2ak + \delta + 3k^2\gamma) \Psi - (3\lambda + 2\sigma) \Psi^3 = 0. \tag{5} \]

As the function \( \Psi \) satisfies both Eqs. (3) and (5), the following constraint condition is given:

\[ \frac{a + 3k\gamma}{3\gamma} = \frac{\omega + ak^2 + \delta k + \gamma k^3}{3(c + 2ak + \delta + 3k^2\gamma)} = \frac{b - k\lambda}{3\lambda + 2\sigma}. \tag{6} \]

Solving for \( k \) and \( c \) in Eq. (6), we have

\[ k = -\frac{(3b\gamma + 2a\sigma + 3a\lambda)}{6\gamma(\lambda + \sigma)}, \tag{7} \]

\[ c = \frac{\gamma(\omega + ak^2 + \delta k + \gamma k^3)}{a + 3k\gamma} - (2ak + \delta + 3k^2). \tag{8} \]

Balancing \( \Psi'' \) and \( \Psi^3 \) in Eq. (3), yields \( m = 1 \).

Consider the trial solutions of Eq. (1) \[4\] to be

\[ \Psi(w) = \sum_{j=1}^{m} \left[ B_j\sinh(w) + A_j\cosh(w) \right]^j + A_0, \tag{9} \]

\[ \Psi(\zeta) = \sum_{j=1}^{m} \left[ \pm iB_j \sech(\zeta) \pm A_j\tanh(\zeta) \right]^j + A_0, \tag{10} \]

\[ \Psi(\zeta) = \sum_{j=1}^{m} \left[ \pm B_j \csch(\zeta) \pm A_j\coth(\zeta) \right]^j + A_0, \tag{11} \]

\[ \Psi(\zeta) = \sum_{j=1}^{m} \left[ \pm B_j \sec(\zeta) + A_j\tan(\zeta) \right]^j + A_0 \tag{12} \]

and

\[ \Psi(\zeta) = \sum_{j=1}^{m} \left[ \pm B_j \csc(\zeta) - A_j \cot(\zeta) \right]^j + A_0. \tag{13} \]

where \( i = \sqrt{-1} \) and \( w' = \sinh(w) \) or \( w' = \cosh(w) \) \[4\].

Substituting Eq. (9) and its second derivative into Eq. (3) with \( m = 1 \), gives a polynomial equation in the power hyperbolic functions. We get a set of algebraic equations by equating each summation of the coefficients of the hyperbolic functions having the same power to zero.
We solve the set of algebraic equations to obtain the values of the parameters involved. To explicitly secure the solutions of Eq. (1), we substitute the values of the parameters into any of Eqs. (10), (11), (12) and (13) with $m = 1$.

**Case-1:** When 

$$A_0 = 0, A_1 = -\sqrt{\frac{k(k(a + k\gamma) + \delta) + \omega}{b - k\lambda}}, \quad B_1 = 0, \quad \mu = \sqrt{-\frac{(k(k(a + k\gamma) + \delta) + \omega)}{2(a + 3k\gamma)}},$$

we get the following dark and singular solitons:

$$\phi_{1,1}(x, t) = \pm \sqrt{\frac{k(k(a + k\gamma) + \delta) + \omega}{b - k\lambda}} \tanh \left[ \mu \left( x - \frac{t^\alpha}{\alpha} \right) \right] e^{i \left( -kx + \omega \frac{t^\alpha}{\alpha} + \theta \right)},$$

and

$$\phi_{1,2}(x, t) = \pm \sqrt{\frac{k(k(a + k\gamma) + \delta) + \omega}{b - k\lambda}} \coth \left[ \mu \left( x - \frac{t^\alpha}{\alpha} \right) \right] e^{i \left( -kx + \omega \frac{t^\alpha}{\alpha} + \theta \right)},$$

respectively, where $k(k(a + k\gamma) + \delta) + \omega > 0$, $b - k\lambda > 0$ and $a + 3k\gamma < 0$ for valid solitons.

3. **Conclusion**

The extended sinh-Gordon equation expansion method is applied in securing the dark and singular optical solitons solutions to the time-fractional Radhakrishnan-Kundu-Lakshmanan equation. We stated the constraint conditions which guarantee the existence of the valid solitons and other solutions. The reported results in this study may be helpful in explaining the physical meaning of the studied nonlinear model and other nonlinear fractional models arising in optical fibers. Based on the computations in this study, it can be seen that the extended sinh-Gordon equation expansion method provides us with powerful mathematical tool for obtaining family of solutions to various nonlinear models arising in mathematical science.

**References**


SENSITIVITY ANALYSIS OF DELAY DIFFERENTIAL EQUATIONS AND OPTIMIZATION PROBLEMS

TAMAZ TADUMADZE

1. ON REPRESENTATION OF THE SENSITIVITY COEFFICIENT

Sensitivity analysis of differential equation consists in finding an analytic relation between solutions of equations with the original and perturbed initial data. It is an important tool for assessing properties of the mathematical models. For example, in an immune model, it allows one to determine dependence of viruses concentrations on the model parameters. Let \( R^n_p \) be the \( n \)-dimensional vector space of points \( x = (x^1, ..., x^n)^T \), where \( T \) denotes transposition; suppose that \( P \subset R^k_p, Z \subset R^n_p \) and \( U \subset R^n_p \) are open sets and \( O = (P, Z)^T = \{ x = (p, z)^T \in R^n_p : p \in P, z \in Z \} \), where \( k + m = n \). Let the \( n \)-dimensional function \( f(t, x, p, z, u) \) be continuous on \( I \times O \times P \times Z \times U \), where \( I = [a, b] \) and continuously differentiable with respect to \( x, p, z \) and \( u \). Furthermore, let \( 0 < \tau_1 < \tau_2, 0 < \sigma_1 < \sigma_2 \) be given numbers and let \( C_p(I_1, R^k_p) \) be the space of continuous functions \( \varphi : I_1 \rightarrow R^k_p \), where \( I_1 = [\hat{t}, b] \), \( \hat{t} = a - \max\{\tau_2, \sigma_2\} \). By \( \Phi = \{ \varphi \in C_p(I_1, R^k_p) : \varphi(t) \in P \} \) and \( G = \{ g \in C_g(I_1, R^m_Z) : g(t) \in Z \} \) we denote sets of initial functions. Let \( \Omega \) be the set of measurable functions \( u(t), t \in I \) satisfying the condition \( clu(I) \subset U \) and it is compact in \( R^n_u \). To each initial data \( \mu = (t_0, \tau, \sigma, p_0, \varphi, g, u) \in \Lambda = (a, b) \times (\tau_1, \tau_2) \times (\sigma_1, \sigma_2) \times P \times \Phi \times G \times \Omega \), we assign the controlled delay functional differential equation

\[
\dot{x}(t) = (p(t), \varphi(t))^T = f(t, x(t), p(t - \tau), z(t - \sigma), u(t))
\]  

with the mixed initial condition

\[
x(t) = (p(t), \varphi(t))^T = (\varphi(t), g(t))^T, t \in [\hat{t}, t_0], x(t_0) = (p(t_0), \varphi(t_0))^T = (p_0, g(t_0))^T.
\]

The condition (1) is said to be a mixed initial condition; it consists of two parts: the first part is \( p(t) = \varphi(t), t \in [\hat{t}, t_0], p(t_0) = p_0 \), the discontinuous part since generally \( p(t_0) \neq \varphi(t_0) \); the second part is \( z(t) = g(t), t \in [\hat{t}, t_0] \), the continuous part since always \( z(t_0) = g(t_0) \).

**Definition 1.** Let \( \mu = (t_0, \tau, \sigma, p_0, \varphi, g, u) \in \Lambda \). A function \( x(t) = x(t; \mu) = (p(t; \mu), \varphi(t; \mu))^T \in O, t \in [\hat{t}, t_1], t_1 \in (t_0, b) \), is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element \( \mu \) and defined on the interval \( [\hat{t}, t_1] \) if it satisfies condition (2) and is absolutely continuous on the interval \( [t_0, t_1] \) and satisfies equation (1) almost everywhere on \( [t_0, t_1] \).

Let \( \mu_0 = (t_{00}, \tau_0, \sigma_0, x_{00}, p_{00}, \varphi_0, g_0, u_0) \in \Lambda \) be a fixed initial data and let \( x(t; \mu_0) \) be the solution corresponding to \( \mu_0 \) and defined on \( [\hat{t}, t_{10}] \), where \( a < t_{00} < t_{10} \), with \( t_{00} + \max\{\tau_2, \sigma_2\} < t_{10} \). Introduce the following notations: \( \delta \mu = (\delta t_0, \delta \tau, \delta \sigma, \delta p_0, \delta \varphi, \delta g, \delta u) \in \Lambda - \mu_0 = \{ \delta \mu = \mu - \mu_0 : \mu \in \Lambda \} \), where \( \delta t_0 = t_0 - t_{00}, \delta \tau = \tau - \tau_0, \delta \sigma = \sigma - \sigma_0, \delta p_0 = p_0 - p_{00}, \delta \varphi = \varphi - \varphi_0, \delta g = g - g_0, \delta u = u - u_0 \). The element \( \delta \mu \) is called variation of the initial data \( \mu_0 \). There
exists number \( \varepsilon_1 > 0 \) such that for any \( \delta \mu \in \Lambda_{\varepsilon_1} = \left\{ \delta \mu \in \Lambda - \mu_0 : \| \delta \mu \| \leq \varepsilon_1 \right\} \) there exists solution \( x(t; \mu_0 + \delta \mu) \) defined on the interval \([\hat{t}, t_{10}]\), here \( \| \delta \mu \| = |\delta \mu_0| + |\delta \tau| + |\delta \sigma| + |\delta p_0| + \| \delta \varphi \|_{L_1} + \| \delta g \|_{L_1} + \| \delta u \|_{L_1} \| \delta \varphi \|_{L_2} = \sup \{ |\delta \varphi(t)| : t \in I_1 \} \).

**Theorem 1.** Let \( \phi_0(t) \) and \( g_0(t) \) be continuously differentiable initial functions. Let the function \( u_0(t) \) be continuous at the points \( t_{10} \) and \( t_{10} + \tau_0 \). Then there exist numbers \( \varepsilon_2 \in (0, \varepsilon_2) \) and \( \delta > 0 \) such that on the interval \([t_{10} - \delta, t_{10} + \delta] \) for arbitrary \( \delta \mu \in \Lambda_{\varepsilon_2} \) we have

\[
x(t; \mu_0 + \delta \mu) = x(t; \mu_0) + \delta x(t; \delta \mu) + o(t; \delta \mu),
\]

where

\[
\delta x(t; \delta \mu) = \left\{ Y(t_{10} + \tau_0; t)f_1(t_{10} + \tau_0) \right\} \delta \tau
\]

\[
- \left\{ Y(t_{10} + \tau_0 + \delta \mu) - Y(t_{10} + \tau_0; t)f_1(t_{10} + \tau_0) \right\} \delta \gamma
\]

\[
- \left\{ \int_{t_{10}}^{t_{10} + \tau_0} Y(\xi; t)f_1(\xi) \right\} \delta \gamma
\]

\[
+ \left\{ \int_{t_{10} + \tau_0}^{t_{10} + \tau_0 + \delta \mu} Y(\xi; t)f_1(\xi) \right\} \delta \gamma
\]

\[
= \left\{ H_{n \times n} \right\} f(\xi; t)\delta \mu
\]

\[
\Theta_{k \times 1} \text{ is the } n \times 1 \text{- zero matrix, } Y(t; s) \text{ is the } n \times n \text{-matrix function satisfying the equation}
\]

\[
Y(\xi; t) = -Y(\xi; t)f_x(\xi) - (Y(\xi; t)f_p(\xi) + \gamma(\xi) + \gamma_0(\xi, t)), \xi \in [t_{10}, t]
\]

and the condition

\[
Y(\xi; t) = \left\{ H_{n \times n} \right\} f(\xi; t)
\]

Here,

\[
\lim_{\| \delta \mu \| \to 0} o(t; \delta \mu)/\| \delta \mu \| = 0 \text{ uniformly for } t \in [t_{10} - \delta, t_{10} + \delta],
\]

\[
H_{n \times n} \text{ is the } n \times n \text{ identity matrix, } f(t) = f(t, x(t), t, x(t), z(t - \tau_0), z(t - \tau_0), z(t - \tau_0), u(t)), f_1(t) = f(t, x(t), t, x(t), z(t - \tau_0), z(t - \tau_0), u(t)), \text{ and } g(t) \text{ is assumed derivative of the function } p(t) \text{ on the set } [t_{10} - \delta, t_{10} + \delta].
\]

The function \( \delta x(t; \delta \mu) \) in the formula (3) is called the coefficient sensitivity of the solution \( x(t; \mu_0), t \in [t_{10} - \delta, t_{10} + \delta] \). The expression (4) is called representation of the sensitivity coefficient.

**2. On Sensitivity of the Minimum of Functional in Optimization Problem**

Let us consider the following optimization problem

\[
\hat{x}(t) = f(t, x(t), t, x(t), z(t - \tau), z(t - \sigma), u(t)), t \in [t_{10}, t_{10}],
\]

\[
x(t) = (\phi_0(t), \gamma_0(t))^T, t \in [\hat{t}, t_{10}], x(t_{10}) = (p_0(t), g_0(t_{10}))^T,
\]

\[
J(w) = q^0(\tau, \sigma, x(t_{11})) + \int_{t_{10}}^{t_{10}} f^0(t, x(t), t, x(t), z(t - \tau), z(t - \sigma), u(t))dt \to \min,
\]

\[
w = (\tau, \sigma, u) \in W = [\tau_1, \tau_2] \times \Theta_1 \times \Theta_2 \times \Omega,
\]

where \( t_{00}, t_{10} \in I \) and \( p_0, p_0 \in P \) are fixed points; \( \phi_0 \in \Phi \) and \( \gamma_0 \) are fixed initial functions; \( x(t) = (p(t), z(t))^T = x(t; w) = (p(t), z(t; w))^T, q^0 \in C_{q} = C_q([\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times O, \Omega) \); \( f^0(t, x(t), p(t), z(t), u(t)) = f^0(t, x(t), p(t), z(t), u(t)) \) is continuous on \([t_{00}, t_{10}] \times O \times P \times \Omega \) and continuously differentiable with respect to \( x, p, z \). By \( W_0 \) we denote the set of \( w \in W \) elements for these there exists the solution \( x(t; w) \) defined on the interval \([t_{00}, t_{10}] \). In that follows it is supposed that \( W_0 \neq \emptyset \).
Theorem 2. There exists an optimal element \( w_0 = (\tau_0, \sigma_0, u_0) \) if the following conditions hold:
1) there exists a compact set \( K_0 \subset O \) such that for an arbitrary \( w \in W_0 \)
\[
x(t; w) \in K_0, t \in [t_{00}, t_{10}];
\]
2) the set
\[
V(t, x, p, z) = \left\{ v = (v^0, v)^T \in \mathbb{R}^{1+n}_v : \exists u \in U, v^0 \geq f^0(t, x, p, z, u), v = f(t, x, p, z, u) \right\}
\]
is convex for all fixed \((t, x) \in [t_{00}, t_{10}] \times K_0 \) and \((p, z)^T \in K_0 \).

Remark 1. Let \( U \) be convex set. Let \( f(t, x, p, z, u) = A(t, x, p, z) + B(t, x, p, z)u \) and let the function \( f^0(t, x, p, z, u) \) be convex in \( u \in U \) then the condition 2) of the Theorem 2 holds.

Theorem 3. Let the conditions 1) and 2) of the Theorem 2 hold. Then for every \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that for every \((p_0, \varphi_0, g_0, q_0^0) \in P \times \Phi \times G \times C_q \) and \( \Phi_\delta(t, x, p, z) = (\psi_\delta, \psi_\delta) \) satisfying the condition \( |p_0 - p_0| + ||\varphi_0 - \varphi_\delta||_1 + ||g_0 - g_\delta||_1 + ||q_0^0 - q_\delta^0|| + ||\Phi_\delta||_1 \leq \delta \) there exists an optimal element \( w_\delta = (\tau_\delta, \sigma_\delta, u_\delta) \) of the perturbed optimization problem
\[
\dot{x}(t) = f(t, x(t), p(t - \tau), z(t - \sigma), u(t)) + \psi_\delta(t, x(t), p(t - \tau), \sigma(t - \sigma)), \\
x(t) = (\varphi_\delta(t), g_\delta(t))^T, t \in [\tau, t_{00}], x(t_{00}) = (p_0, g_\delta(t_{00}))^T \\
J(w; \delta) = q_\delta^0(\tau, \sigma, x(t_1)) + \int_{t_{00}}^{t_{10}} \left[ f^0(t, x(t), p(t - \tau), z(t - \sigma), u(t)) + \psi_\delta^0(t, x(t), p(t - \tau), z(t - \sigma)) \right] dt \rightarrow \min
\]
and
\[
|J(w_0) - J(w_\delta; \delta)| < \varepsilon.
\]
Here, the function \( \Psi_\delta \) satisfies the standard conditions and
\[
\int_{t_{00}}^{t_{10}} \sup \left\{ |\Psi_\delta(t, x, p, z)| + |\Psi_\delta(t, ..)| + |\Psi_\delta(t, ..)| + |\Psi_\delta(t, ..)| : x \in K_1, (p, z)^T \in K_1 \right\} dt \leq \text{const},
\]
where \( K_1 \subset O \) is a compact set containing a neighborhood of \( K_0 \):
\[
\Psi_\delta = \frac{\partial}{\partial x} \Psi_\delta, ||q_0 - q_\delta|| = \sup \left\{ |q_\delta(\tau, \sigma, x) - q_\delta(\tau, \sigma, x)| : (\tau, \sigma, x) \in [\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times K_1 \right\},
\]
\[
||\Psi_\delta||_1 = \sup \left\{ \int_{s_1}^{s_2} \Psi_\delta(t, x, p, z) dt : (s_1, s_2, x) \in [t_{00}, t_{10}] \times [t_{00}, t_{10}] \times K_1, (p, z)^T \in K_1 \right\}.
\]

Theorems 1-3 are proved by schemes given in [1],[2].

Keywords: Sensitivity analysis, delay differential equation, optimization.

AMS Subject Classification: 34K27, 34K99, 49J21, 49K40.

References

ON TOPOLOGICALLY UNIFORMLY TRANSITIVE ACTIONS OF LOCALLY COMPACT UNIMODULAR AMENABLE GROUPS

A.T. TAGI-ZADEH

1 Baku State University, Institute of Applied Mathematics, Baku, Azerbaijan.
e-mail: azdtagi@rambler.ru

Abstract. Initiated by Kolmogorov's ideas, the entropy theory of dynamical systems is actively developing at the present time. The measure theoretical entropy for the actions of the LCUA groups was introduced and studied in the works of D. Ornstein and B. Weiss. Topological entropy, as well as an alternative definition of measure theoretical entropy for the actions of LCUA groups, was introduced by the author. In this paper we introduce the notion of topologically uniformly transitive actions of LCUA groups and prove some properties of the topological and measure theoretical entropies of such actions.

Keywords: Ergodic theory, dynamical systems, groups actions, amenable groups, measure-theoretical entropy, topological entropy.

AMS Subject Classification: 37A05, 43A07, 37A35, 37B40, 16W22.

1. Introduction:

Initiated by Kolmogorov’s ideas, the entropy theory of dynamical systems is actively developing at the present time and is finding increasing application not only in the diverse areas of mathematics, but also in many applied problems. In [1] a variational principle for the topological pressure of countable amenable groups transformations is proved, and the existence of a Gibbs measure of maximum pressure is established for lattice systems. The measure theoretical entropy for the actions of the LCUA groups was introduced and studied in the works of D. Ornstein and B. Weis. Topological entropy, as well as an alternative definition of measure theoretical entropy for the actions of LCUA groups, was introduced by the author in [2], [3]. In [2] a variational principle for topological entropy was proved for the actions of LCUA groups with lattices. In this paper we introduce the notion of topologically uniformly transitive actions of LCUA groups and prove some properties of the topological and measure theoretical entropies of such actions. The approaches underlying these studies are, in my opinion, the natural development of the corresponding ideas of R. Bowen and P. Walters.

Necessary concepts and definitions: Let $T$ be the continuous action of the LCUA group $(G, \lambda)$ in the compact metric space $(X, \lambda)$, where $\lambda$ is a measure of Haar, and, $\mathcal{FAS}(G)$ is the set of averaging Folner sequences - the set of sequences $\{F_n\}_{1}^{\infty}$ of compact subsets of $G$, such that for any compact set $K \subset G$ the following equality holds:

$$\lim_{n \to \infty} (\lambda(F_n))^{-1} \lambda(K \cdot F_n \Delta F_n \cdot K) = 0.$$
For a compact set $F \subseteq G$ and $A = \left( C_A^1, \ldots, C_n^A \right) -$open covering of $X$ a finite set $A$ is $(F, A)$-separated if for any $x, y \in A$ there is $g \in G$ such that:

$$\{i | 1 \leq i \leq n, T_x^i \in C_A^1 \} \cap \{i | 1 \leq i \leq n, T_y^j \in C_A^1 \} = \emptyset$$

We now define for $A$-a finite covering of $X$ and a compact set $F \subseteq G$:

$$S(F, A) = \max \{ \text{card } s \subseteq X, s - (F, A) - \text{separated} \};$$

for $F = \{F_n\}_{n=1}^\infty$, $F \in FAS(G)$

$$h^F(T, A) = \lim_{n \to \infty} (\lambda(F_n))^{-1} \log S(F_n, A);$$

$$h^F(T) = \sup \{ h^F(T, A) | A - \text{is an open covering of } X \}$$

**Theorem 1:** The topological entropy of the action of LCUA group on a compact metric space does not depend on the choice of the averaging Følner sequence.

**Definition 1.** The action $T$ of LCUA group $G$ on a compact metric space $X$ is said to be topologically uniformly transitive if for any $\varepsilon > 0$ there exist a finite open covering $A$ with $\text{diam}A < \varepsilon$ and a compact set $K \in K(G)$, such that for any compact sets $F, H \in K(G)$ the inequality holds

$$S(F \cup H, A) \geq S(F \setminus H, A) \cdot S(H \setminus K \cdot (F \setminus H), A) \tag{1}$$

**Remark 1.** In particular, Bernoulli shifts and "solid kernels" systems satisfy this property. For the metric entropy $h^F(T) \ (2,3)$ of the action of LCUA group, the following is true:

**Theorem 2.** Let $T$ be a topologically uniformly transitive action of the LCUA group $G$ in a compact metric space $X$. Then for an arbitrary sequence $F = \{F_n\}_{n=1}^\infty \in FAS(G)$

$$\sup_{\mu \in \mathfrak{M}(X, T)} h^F(T) \geq h^F(T)$$

**Proof.** We fix a Følner sequence $f = \{F_n\}_{n=1}^\infty$ and an arbitrary $\varepsilon > 0$. Let $A$ be a covering of $X$ such that inequality holds:

$$h^F(T) < h^F(T, A) + \varepsilon.$$ 

Further, without loss of generality, we can assume that

$$h^F(T, A) = \lim_{n \to \infty} |F_n|^{-1} \ln S(F_n, A).$$

We choose $N_0 \in \mathbb{N}$ such that for any $n > N_0$ we have the inequality

$$(\lambda(F_n))^{-1} \cdot |\ln S(F_n, A) - h^F(T, A)| < \varepsilon.$$ 

We fix then $\{S^*(F_n, A)\}_{n=1}^\infty$ - the sequence of $(F_n, A)$-separated sets for which $\text{card}(S^*(F_n, A)) = S(F_n, A)$. For $n \in \mathbb{N}$ we now define the sequence $\{\sigma_n\}$ of probability measures on $X$ as follows:

$$\sigma_n(\{x\}) = \begin{cases} 
\text{card}(S^*(F_n, A))^{-1}, & \text{for } x \in S^*(F_n, A) \\
0, & \text{for } x \notin S^*(F_n, A)
\end{cases}$$

Further, we define a sequence of measures $\{\mu_n\}_{n=1}^\infty$ by equality

$$\{\mu_n\}(\cdot) = |F_n|^{-1} \int_{F_n} \sigma_n \left( T^{g^{-1}} \right) d\lambda(g).$$

Let $\mu$ be a weak limit point of the sequence $\{\mu_n\}_{n=1}^\infty$. We choose a finite Borel partition $\zeta \in P(X)$ such that $\text{diam} \zeta < \frac{1}{4} \lambda(A)$ and $\mu(\partial \zeta) = 0$. Then we choose a sequence $K = \{K_n\}_{n=1}^\infty \in K_1(\lambda(A)) (\mathcal{F}).$

By the choice of the measure $\sigma_n$ for $\alpha \in \zeta^K$ we have

$$\sigma_n(\alpha) = (S(F_n, A))^{-1} \cdot \text{card}(\alpha \cap S^*(F_n, A)) \tag{2}.$$
We choose and fix $K \in K(G)$ so that the inequality holds:

$$S(F_n, A) \geq S(F_n \setminus F_k, A) \cdot S(F_k \setminus K(F_n \setminus F_k), A)$$

and from (1) we have

$$\sigma_n(\alpha) \leq (S(F_n \setminus F_k, A) \cdot S(F_k \setminus K(F_n \setminus F_k), A))^{-1} \cdot card(\alpha \cap S^*(F_n, A)).$$

On the other hand, since $x, y \in \alpha$, then for an arbitrary $g \in F_k$ and for $h \in K_n$ such that $\rho_T(h, g) \leq 1/4 \lambda(A)$ we have

$$\rho(T^g x, T^g y) \leq \rho(T^g x, T^h x) + \rho(T^g y, T^h y) + \rho(T^h x, T^h y) \leq \frac{1}{4} \lambda(A) + \frac{1}{4} \lambda(A) + \text{diam} \zeta < \lambda(A)$$

and hence the set $S^*(F_n, A) \cap \alpha$ is $(F_n \setminus F_k, A)$ separated and the inequality

$$\text{card}(\alpha \cap S^*(F_n, A)) \leq S(F_n \setminus F_k, A)$$

is true.

Further, from the method of estimating the quantity $\sigma_n(\alpha)$ we obtain the inequality

$$|F_k|^{-1} H_{\sigma_n}(T^g \zeta K_m) \geq (1 - \varepsilon)(h^F(T) - 2\varepsilon)$$

for any $g \in F_n$.

Hence, by the convexity of the entropy function $-t \ln t$ and the choice of the measure $\{\mu_n\}$ the assertion of the theorem follows.

References


THE FREQUENCY HIGH ACCURACY ALGORITHM FOR THE SOLUTION OF CONTINUOUS LINEAR-QUADRATIC OPTIMAL SYNTHESIS PROBLEM BY OUTPUT VARIABLE

N.I. VELIEVA

1 Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan

1. INTRODUCTION

As is known [4], the solution of the problem of synthesis of optimal regulators is constructed with the help of parameterization of independent matrices, encompassing a more general range of problems [4] than [6, 8], which are aimed only at cases when all the coordinates of an object are measured. In [1, 2, 4, 8, ] a special case is considered, i.e. the parts of the object’s phase coordinates are measured and, introducing a small parameter, it is assumed that all the coordinates of the object with noise interference are measured. Further, using the approach to zero of the small parameter are obtained asymptotic formulas for regulators for part of the object’s phase coordinates. It also shows that when these processes are performed, appropriate transformations are required which, for numerical high-dimensional realizations, face great difficulties.

However, in [3, 5] there is no computational algorithm for the solution of this problem, which does not allow expanding these results even for the development of high accuracy algorithms. Therefore, in this paper, an algorithm for solving the problem of optimal synthesis by output in a more general case is given so that it is possible to create a high-precision algorithm for the optimal controller using the procedures of the Symbolic Tollbox of the MatLab package. The results are illustrated by examples [5, 7], where in [7] a regulator other than [7] is obtained, which ensures the asymptotic stability of the closed system and the value of the functional is much less than in [7]. In the second example [5], using the proposed algorithm, the same expressions for regulators were obtained.

2. STATEMENT OF THE PROBLEM

Movement of the object described by the linear polynomial operator in the frequency domain

\[ Py_1 = Mu_2 + \psi. \] (1)

Vector of observed values is expressed in the following form

\[ \Sigma_0 y_2 = \Sigma_1 y_1 + \phi, \] (2)

where \( y_2 \) - is a observable vector, \( y_1 \) - is a vector coordinates of the object to be stabilized, \( u_2 \) - is a control vector, \( \psi, \phi \) - is a vectors of stationary random external influences and the measurement noise is a stationary random processes with fractional rational matrices of spectral densities \( S_\psi, S_\phi \). \( P, M, \Sigma_0, \Sigma_1 \) polynomial matrices are functions of the parameter \( s \) Laplace transform and have the corresponding dimensions. The problem consists in finding the equation of the regulator in the form

\[ W_0 u_2 = W_1 y_2, \] (3)

which minimization the functional

\[ I = \langle y_1^T R y_1 \rangle + \langle u_2^T C u_2 \rangle \] (4)

where \( R, C \) are symmetric positive definite matrices.
provided asymptotic stability of the closed system (1) - (3), i.e. the roots of the matrix of the closed system (1) - (3)

\[
\begin{pmatrix}
P & -M & 0 \\
\Sigma_1 & 0 & -\Sigma_0 \\
0 & -W_0 & W_1
\end{pmatrix}
\]

lie on the left half-plane. Here \(W_0, W_1\) - the required matrices of appropriate dimensions, the elements of which are operator polynomials, \(R, C\) - weight matrices, \(\langle .. \rangle\) - symbol of mathematical expectation.

Accepting below the following designation

\[G_{11} = P^{-1}\theta_1, \quad \theta_1 = [E 0],\]
\[G_{12} = P^{-1}M,\]
\[G_{21} = \Sigma_0^{-1}(\Sigma_1P^{-1}\theta_1 + \theta_2), \quad \theta_2 = [0 E],\]
\[G_{22} = \Sigma_0^{-1}\Sigma_1P^{-1}M, \quad u_1 = [\psi \phi]',\]

where \(E\) unit matrix of the corresponding dimension. Then we write the system (1) - (3) in the form

\[
y_1 = G_{11}u_1 + G_{12}u_2, \\
y_2 = G_{21}u_1 + G_{22}u_2, \\
u_2 = -Wy_2, \quad W = W_0^{-1}W_1.
\]

Then the matrix of the transfer function of the corresponding regulator is defined in the following form

\[W = (v_r - QN_l)^{-1}(u_r + QD_l).\]

Here \(D_l, N_l\) - the result of the left MFD representation of the matrix \(G_{22}\)

\[G_{22} = D_l^{-1}N_l,\]

\(D_r, N_r\) - the result of the right MFD representation of the matrix \(G_{22}\)

\[G_{22} = N_rD_r^{-1},\]

\(v_r, u_r\) - solution of the following matrix Diophantine equation

\[u_rN_r + v_rD_r = E.\]

with this choice \(v_r, u_r\) the matrix of the transfer function of the corresponding regulator (6) stabilization of the system (1).

Calculating the left representation of the matrix \(\Sigma_1P^{-1} = \tau^{-1}\gamma, \quad \gamma M, \quad D_l = \tau\Sigma_0.\)

We introduce the notation:

\[Z = \begin{bmatrix} v_r & u_r \\ N_l & -D_l \end{bmatrix}, \quad Z^{-1} = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}, Z^{-1} = E.\]

and expressions for the vector of the minimized error

\[e = \begin{bmatrix} R^{1/2}y_1 \\ C^{1/2}u_2 \end{bmatrix} = \begin{bmatrix} R^{1/2}P^{-1}[\theta_1 - M(\theta_1Q + \theta_1)(\gamma\theta_1 + \tau\theta_2)] \\ C^{1/2}(\theta_1Q + \theta_1)(\gamma\theta_1 + \tau\theta_2) \end{bmatrix} u_1,\]

where \(R^{1/2}\) - the square root of the matrices \(R, C^{1/2}\) - the square root of the matrices \(C.\)

Taking this into account in the functional (4), we obtain

\[J = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \text{Sp}(e^*e)ds, \quad s = i\omega.\]
denote
\[ \bar{\theta}_1 = \begin{bmatrix} S^{1/2} \\ 0 \end{bmatrix}, \quad \bar{\theta}_2 = \begin{bmatrix} 0 & S^{1/2} \end{bmatrix}, \quad S^{1/2} S^{1/2} = S, \quad S^{1/2} S^{1/2} \ast = S, \quad S^{1/2} S^{1/2} \ast = S, \]
then we get
\[ e = [T_1 - T_2 Q T_3] W, \]
where
\[ T_1 = \begin{pmatrix} R^{1/2} P^{-1} [(E - M \theta_{12} \gamma) \bar{\theta}_1 - M \theta_{12} \tau \bar{\theta}_2] \\ -C^{1/2} \theta_{12} (\gamma \bar{\theta}_1 + \tau \bar{\theta}_2) \end{pmatrix}, \]
\[ T_2 = \begin{pmatrix} R^{1/2} P^{-1} M \theta_{11} \\ C^{1/2} \theta_{11} \end{pmatrix}, \]
\[ T_3 = \gamma \bar{\theta}_1 + \tau \bar{\theta}_2. \]
The initial problem is equivalent to the problem of minimizing the norm of matrices \( T_1 - T_2 Q T_3 \) on the set of matrices \( Q \), has no poles in the right half. To find the matrix \( Q \) introduce the following notation:
\[ H^{\ast} H = T_2 T_2, \quad D D^{\ast} = T_3 T_3^{\ast}, \quad L_0 + L_{+} + L_{-} = H^{-1} T_2, T_1 T_3, D^{-1}. \]
\( L_{0} \)- constant or polynomial of \( s \), \( L_{+} \)- proper fraction with poles in the left, \( L_{-} \)- with poles in the right half-plane. Then
\[ Q = H^{-1} (L_0 + L_+) D^{-1}. \]
The optimal regulator has the form
\[ W = (v_r - Q N) \ast^{-1} (u_r + Q D), \]
The solution to this problem is reduced to (7) - (16). Here the main place is taken by the factorization of polynomials and the separation of fractional-rational expressions, the solution of Diophantine equations, and the MFD representation of transfer matrices. For factorization of polynomials [1, 2] and separation of fractional-rational expressions, a highly accurate algorithm [1, 2] is developed, which reduces to the solution of the algebraic Riccati equation. To solve Diophantine equations (9) the method of exhaustive search all possible values of the variables in the equation. This method is easily implemented in the Symbolic Toolbox of the Matlab package.

**Keywords:** Algorithm of parametrization, matrix fraction decomposition, solution of the Diophantine equation, factorization of matrix polynomials

**AMS Subject Classification:** 49J15, 49J35.

**References**

THE GOURSAT PROBLEM FOR THE PSEUDOPARABOLIC EQUATION
OF THE THIRD ORDER WITH SINGULAR COEFFICIENTS

M.H. YAGUBOV, SH.SH. YUSUBOV

1Department of Mechanics and Mathematics, Baku State University, Baku, Azerbaijan
e-mail: yusubov_sh@mail.ru

Abstract. We consider the Goursat problem for the third-order pseudoparabolic equation
with singular coefficients. A weight space is introduced and using the integral representation of
functions in this space a homeomorphism between definite pairs of Banach spaces is established.
The uniqueness of the solution is proved for the considered problem.

Keywords: Pseudoparabolic equation of the third order, Goursat problem, singular coefficients.

AMS Subject Classification: 35L30, 35L81.

In the domain $G = \{(t, x) : t \in G_1 = (t_0, t_1), x \in G_2 = (x_0, x_1)\}$ we consider the hyperbolic
equation of the third order with singular coefficients

$$
(1_{12} u)(t, x) \equiv D_t D_x^2 u + \frac{a_{02}(t, x)}{(t - t_0)^{\alpha_1}} D_x^2 u + \frac{a_{11}(t, x)}{(x - x_0)^{\alpha_2}} D_t D_x u + \frac{a_{10}(t, x)}{(x - x_0)^{\alpha_2}} D_t u + \\
\frac{a_{01}(t, x)}{(x - x_0)^{\alpha_1}} D_x u + \frac{a_{00}(t, x)}{(t - t_0)^{\alpha_1}(x - x_0)^{\alpha_2}} u = \varphi_{12}(t, x),
$$

At the present time, boundary value problems for the third-order hyperbolic equations that
are of great theoretical and practical interest are actively studied.

We note that for $\alpha_1 = \alpha_2 = 0$ equation (1) and its particular cases for various local and
nonlocal boundary conditions are investigated, for example, in [1-8]. In these papers, depending
on the smoothness of the coefficients and the right-hand side of equation (1), the solution is
sought in the classes $C^{(1,2)}(G)$ or $W_p^{(1,2)}(G)$.

If in equation (1) $a_{ij}(t, x) = 0$, $i = 0, 1, j = 0, 2$, $\alpha_1^2 + \alpha_2^2 > 0$,

$$
\varphi_{12}(t, x) = (t - t_0)^{-\frac{1}{2}}(x - x_0)^{-\frac{1}{2}} \ln \frac{t - t_0}{2} \left| \ln \frac{t - t_0}{2} \right|^{-\frac{2}{p}} \ln \frac{x - x_0}{2} \left| \ln \frac{x - x_0}{2} \right|^{-\frac{2}{p}} ,
$$

$(t, x) \in G = \{(t, x) : t \in G_1 = (t_0, t_0 + 1), x \in G_2 = (x_0, x_0 + 1)\}$,

then $D_t D_x^2 u$ does not belong to the space $L_p(G)$ and therefore equation (1) does not have a
solution in the space $W_p^{(1,2)}(G)$. It is well known that the study of boundary value problems for
differential equations with singular coefficients naturally leads to the introduction of "weighted"
functional classes. Therefore, it is of interest to introduce such a space that equation (1) has a
solution in this space and for $\alpha_1^2 + \alpha_2^2 \geq 0$. 376
In this paper we introduce the weighted spaces $W_{p,\alpha}^{(1,2)}(G)$, $H_{p,\alpha}^{(1,2)}$ and by the help of the integral representation from $W_{p,\alpha}^{(1,2)}(G)$ the isomorphism of the spaces $W_{p,\alpha}^{(1,2)}(G)$, and $H_{p,\alpha}^{(1,2)}$ is proved. Further, using this, we prove the solvability of certain boundary value problems for equation (1) in the space $W_{p,\alpha}^{(1,2)}(G)$.

Consider equation (1) under the following conditions

$$(l_{02}u)(x) = D_x^2 u(t_0, x) = \frac{\varphi_{02}(x)}{(x - x_0)^{\alpha_2}}, \quad x \in G_2,$$

$$(l_{11}u)(t) = D_t D_x u(t, x_0) = \frac{\varphi_{11}(t)}{(t - t_0)^{\alpha_1}}, \quad t \in G_1,$$

$$(l_{10}u)(t) = D_t u(t, x_0) = \frac{\varphi_{10}(t)}{(t - t_0)^{\alpha_1}}, \quad t \in G_1,$$

$$l_{01}u = D_x u(t_0, x_0) = \varphi_{01},$$

$$l_{00}u = u(t_0, x_0) = \varphi_{00}.$$  \hspace{1cm} (2)

Here $u(t, x)$ -- is a sought function, $D_s^k = \frac{\partial^k}{\partial x^k}$ is an operator of the generalized differentiation in the Sobolev sense, $a_{ij}(t, x), i = 0, 1, j = 0, 2, i + j < 3$ are measurable on $G$ functions satisfying conditions $a_{00}(t, x), a_{01}(t, x) \in L_p(G)$, $a_{11}(t, x) \in L_{\infty,p}(G)$, $j = 0, 1, a_{02}(t, x) \in L_{p,\infty}(G)$, and by the help of the

We note that the weighted space $W_{p,\alpha}^{(1,2)}(G)$ with the given norm is Banach and the function $u(t, x)$ of this space has an integral representation

$$u(t, x) = (Qb)(t, x) \equiv b_{00} + (x - x_0)b_{01} + \int_{t_0}^t \frac{1}{(\tau - t_0)^{\alpha_1}}(b_{10}(\tau) + (x - x_0)b_{11}(\tau))d\tau +$$
\[
\int_{x_0}^{x} \frac{x-s}{(s-x_0)^{\alpha_2}} b_{02}(s) ds + \int_{t_0}^{t} \frac{x-s}{(\tau-t_0)^{\alpha_1}(s-x_0)^{\alpha_2}} b_{12}(\tau, s) d\tau ds,
\]

where

\[
b = \left( \frac{b_{12}(t, x)}{(t-t_0)^{\alpha_1}(x-x_0)^{\alpha_2}}, \frac{b_{02}(x)}{(x-x_0)^{\alpha_2}}, \frac{b_{11}(t)}{(t-t_0)^{\alpha_1}}, \frac{b_{10}(t)}{(t-t_0)^{\alpha_1}}, b_{01}, b_{00} \right) \in H_{p,\alpha}^{(1,2)},
\]

\[
H_{p,\alpha}^{(1,2)} = L_{p,\alpha}(G) \times L_{p,\alpha_2}(G_2) \times L_{p,\alpha_1}(G_1) \times R \times R, \quad b_{12} \in L_p(G), \quad b_{02} \in L_{p}(G_1), \quad b_{11}, b_{10} \in L_p(G_1), \quad b_{01}, b_{00} \in R.
\]

The norm in the space \(H_{p,\alpha}^{(1,2)}\) can be defined in a natural way by means of equality

\[
\|b\|_{H_{p,\alpha}^{(1,2)}} = \left\| \left( \frac{b_{12}}{(t-t_0)^{\alpha_1}(x-x_0)^{\alpha_2}}, \frac{b_{02}}{(x-x_0)^{\alpha_2}}, \frac{b_{11}}{(t-t_0)^{\alpha_1}}, \frac{b_{10}}{(t-t_0)^{\alpha_1}}, b_{01}, b_{00} \right) \right\|_{p,\alpha,G} + \left\| \left( \frac{b_{01}}{(t-t_0)^{\alpha_1}}, \frac{b_{00}}{(t-t_0)^{\alpha_1}} \right) \right\|_{p,\alpha,G} + \|b_{01}\| + \|b_{00}\|.
\]

It follows from the representations (3) that any function \(u \in W_{p,\alpha}^{(1,2)}(G)\) has traces \(D_{2}^{2}u(t_0, x)\), \(D_{1}^{2}u(t_0, x_0)\), \(i = 0, 1\), \(D_{x}^{2}u(t_0, x_0)\), \(i = 0, 1\) and the operations of taking these traces are continuous from \(W_{p,\alpha}^{(1,2)}(G)\) to \(L_{p,\alpha_2}(G_2), L_{p,\alpha_1}(G_1), R\), respectively. For these traces, we also have equalities

\[
D_{2}^{2}u(t_0, x) = \frac{b_{02}(x)}{(x-x_0)^{\alpha_2}}, \quad D_{1}^{2}D_{x}^{2}u(t_0, x_0) = \frac{b_{11}(t)}{(t-t_0)^{\alpha_1}}, \quad i = 0, 1,
\]

\[
D_{x}^{2}u(t_0, x_0) = b_{0i}, \quad i = 0, 1.
\]

**Theorem 1.** There exist positive constants \(C_1\) and \(C_2\) such that

\[
C_1 \|b\|_{H_{p,\alpha}^{(1,2)}} \leq \|Qb\|_{W_{p,\alpha}^{(1,2)}(G)} \leq C_2 \|b\|_{H_{p,\alpha}^{(1,2)}}.
\]

We get from these inequalities that the operator \(Q\) sets isomorphism between the spaces \(H_{p,\alpha}^{(1,2)}\) and \(W_{p,\alpha}^{(1,2)}(G)\). Therefore the spaces \(W_{p,\alpha}^{(1,2)}(G)\) and \(H_{p,\alpha}^{(1,2)}\) may be identified in the sense of isomorphism.

**Theorem 2.** For any \(\varphi = \left( \frac{\varphi_{12}(t, x)}{(t-t_0)^{\alpha_1}(x-x_0)^{\alpha_2}}, \frac{\varphi_{02}(x)}{(x-x_0)^{\alpha_2}}, \frac{\varphi_{11}(t)}{(t-t_0)^{\alpha_1}}, \frac{\varphi_{10}(t)}{(t-t_0)^{\alpha_1}}, \varphi_{01}, \varphi_{00} \right) \in H_{p,\alpha}^{(1,2)}\) problem (1), (2) has a unique solution.

**References**


ON THE DESIGN OF INTERNAL MODEL GENERATORS FOR REPETITIVE CONTROLLERS

VEYSEL YÜCESOY\textsuperscript{1,2}, HITAY ÖZBAY\textsuperscript{2}

\textsuperscript{1}ASELSAN Research Center, 06370, Yenimahalle, Ankara, Turkiye
\textsuperscript{2}Bilkent University, 06800, Bilkent, Ankara, Turkiye
e-mail: vyucesoy@aselsan.com.tr, hitay@bilkent.edu.tr

ABSTRACT. This study revisits the problem of designing internal model generators for repetitive control problems. We showed that it is possible to design internal model generators which have finitely many poles in the closed right half plane including the imaginary axis via Nevanlinna-Pick interpolation. The proposed method is also illustrated on a numerical example.

Keywords: Nevanlinna-Pick interpolation, robust control, repetitive control, time delay.

AMS Subject Classification: 30E05, 47N70, 93C05, 93C80.

1. Introduction

In the literature, repetitive controllers are used to track periodic reference signals, \cite{1}. A repetitive controller $C = C_oC_u$ consists of two cascade transfer functions, one of which is an internal model generator $C_u$ and the other one is a compensator to stabilize the feedback loop. Typically, $C_u$ is in the form $(1 - e^{-Ls})^{-1}$ in order to track periodic reference signals whose periods are $L > 0$. This idea comes from the internal model principle, which states that a copy of the reference signal should appear within the closed loop for perfect tracking, \cite{2}. However, this unrealistic demand for tracking performance comes at the expense of closed loop robustness and restrictions on the applications, see for example \cite{6}. In order to handle these problems, a filter $q(s)$ is added in cascade with the time delay term within the internal model generator $C_u$, see \cite{3} for details. With this modification, the resulting structure of $C_u$ becomes

\begin{equation}
C_u(s) = \frac{1}{1 - q(s)e^{-Ls}}.
\end{equation}

Note that, a strictly proper, stable and low-pass choice of $q(s)$ yields finitely many unstable poles for $C_u$ in the open right half plane, hence relaxes the issues about robustness and restrictions on the applications. In addition to this, for repetitive performance, $C_u$ must have at least some finitely many poles on the imaginary axis.

2. Design of Repetitive Filters

Let us assume that we would like to have $(n + 1)$ poles of $C_u$ (i.e. $\{p_0, p_1, \ldots, p_n\}$) on the imaginary axis. For repetitive performance, these poles should be placed as follows:

- $p_0 = 0$
- $p_{2k-1} = j2\pi k/L$ for $k = \{1, 2, \ldots, n/2\}$
- $p_{2k} = -j2\pi k/L$ for $k = \{1, 2, \ldots, n/2\}$.
Note that \( n \) has to be an even positive integer due to conjugate symmetry of \( p_k \) for \( k = 1, 2, \ldots, n \). It is easy to verify that \( C_u \) has a pole at any \( p_k \) if \( q(p_k) = 1 \). In light of this interpretation, the problem reduces to finding a strictly proper and stable transfer function \( q(s) \) which is equal to 1 at \( p_k \) and has a low pass characteristic.

Let us recall the well-known Nevanlinna-Pick interpolation problem (NPIP): Given the interpolation data \( \mathfrak{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \) and \( \mathfrak{B} = \{\beta_1, \beta_2, \ldots, \beta_m\} \) where \( \alpha_i \in \mathbb{C}_+ \) and \( \beta_i \in \mathbb{C} \) for all \( i \in \{1, 2, \ldots, m\} \), the NPIP is to find a stable transfer function \( F \) such that \( F(\alpha_i) = \beta_i \) and \( \|F\|_\infty < \gamma \) for the smallest possible \( \gamma = \gamma_{\text{opt}} \). Clearly set \( \mathfrak{A} \) has distinct elements and in order to restrict the solutions to transfer functions having real coefficients, we furthermore assume that the set \( \mathfrak{A} \) and the set \( \mathfrak{B} \) are conjugate symmetric, i.e. if \( \alpha_i \in \mathfrak{A} \) and \( \beta_j \in \mathfrak{B} \) satisfies \( F(\alpha_i) = \beta_j \) then \( \bar{\alpha}_i \in \mathfrak{A} \) and \( \bar{\beta}_j \in \mathfrak{B} \) and they satisfy \( F(\bar{\alpha}_i) = \bar{\beta}_j \). It is well-known that NPIP is feasible if and only if the associated Pick matrix is positive semi-definite, see [4] for further details and parametrization of all suboptimal solutions. Additionally, it has been proven that the optimal solution of NPIP is an inner function [5] and it is directly possible to calculate \( \gamma_{\text{opt}} \) from the interpolation data. Recently a direct computation procedure is derived for NPIP, see [7] for details.

**Proposition 1.** Given a low pass, stable and strictly proper transfer function \( K(s) \), it is possible to use \( q(s) = F(s)K(s) \) as the filter in front of the delay term in the denominator of \( C_u \) to place \( n + 1 \) poles of \( C_u \) on the imaginary axis and to have finitely many right half plane poles if it is possible to find \( F \) such that

\[
\begin{align*}
F & \in \mathcal{H}_\infty \\
F(p_k) & = K^{-1}(p_k) \text{ for all } k \in \{0, 1, \ldots, n\} \\
\|F\|_\infty & \leq \gamma_{\text{opt}}
\end{align*}
\]

**Proof.** We know that finding such \( F \) is a NPIP with boundary interpolation conditions (i.e. \( p_k \notin \mathbb{C}_+ \)). Hence, it is easy to see that \( q = FK \) has a low pass characteristic, it is stable and strictly proper as \( K \). Finally, we can also verify that \( q(p_k) = 1 \) for repetitive performance due to interpolation conditions on \( F \).

To the best of our knowledge, there does not exist a simple algorithm (which does not use any conformal map) to find optimal solution of NPIP with boundary conditions, however, there are ways to handle such boundary conditions. One of these ways is to shift the interpolation data to right half plane by some amount \( \varepsilon \) and solve the problem in a shifted space to back transform the shifted optimal solution to original space, see [5] for other methods to handle boundary conditions. In the light of this interpretation, we can write the NPIP associated to \( F \) as follows: Find \( \hat{F} \) for some small number \( \varepsilon \) such that

\[
\begin{align*}
\hat{F} & \in \mathcal{H}_\infty \\
\hat{F}(\varepsilon + p_k) & = K^{-1}(p_k) \text{ for all } k \in \{0, 1, \ldots, n\} \\
\|\hat{F}\|_\infty & \leq \hat{\gamma}_{\text{opt}}
\end{align*}
\]

If such \( \hat{F} \) exist, then \( F(s) = \hat{F}(s + \varepsilon) \). Note that \( \gamma_{\text{opt}} = \hat{\gamma}_{\text{opt}} \) for small values of \( \varepsilon \).

3. Numerical Example

Let us design \( q(s) \) for \( L = 1, n = 4, K(s) = (s + 1)^{-1} \) and \( \varepsilon = 0.1 \). For interpolation data \( \{\varepsilon, 1\}, (\varepsilon + j2\pi, 1 + j2\pi), (\varepsilon - j2\pi, 1 - j2\pi), (\varepsilon + j4\pi, 1 + j4\pi), (\varepsilon - j4\pi, 1 - j4\pi) \}, \) by using the algorithm of [7], \( \hat{F} \) is calculated as follows for \( \hat{\gamma}_{\text{opt}} = 12.713 \)

\[
\hat{F}(s) = \frac{12.713(s - 11.06)(s - 0.1174)(s^2 - 0.1375s + 40.66)}{(s + 11.06)(s + 0.1174)(s^2 + 0.1375s + 40.66)}
\]

and the associated \( q(s) \) is given as

\[
q(s) = \frac{12.713(s - 10.96)(s - 0.01743)(s^2 + 0.06252s + 40.65)}{(s + 1)(s + 11.16)(s + 0.2174)(s^2 + 0.3375s + 40.68)}.
\]
One can easily verify that $q(s)$ has a low-pass characteristics. Let us check the root locations of $1 - q(s)e^{-s}$ to verify our design. Fig. 1 shows the root locations of $1 - q(s)e^{-s}$ and it is clear that 5 roots are located on the imaginary axis at points $\{0, \pm 2\pi, \pm 4\pi\}$. Furthermore there are only 4 roots in the open right half plane and all other roots are in the left half plane. One can now easily design a stabilizing controller for $(1 - q(s)e^{-s})^{-1}P_o(s)$ for any finite dimensional $P_o(s)$ by using the methods of [5]. Although extra unstable roots appear in $(1 + q(s)e^{-Ls})^{-1}$ using this approach, it is still possible to design a stabilizer. Extra conditions for avoiding these open right half plane poles will be discussed at another forthcoming publication.

4. Conclusion

This paper revisits the problem of designing internal model generators for repetitive controllers. For the robustness of the closed loop and practical applications it is desirable to find internal model generators having finitely many right half plane poles in addition to imaginary axis poles for repetitive performance. We have proposed an interpolation based method to design strictly proper, stable and low pass filters which can be used to design such internal model generators. The proposed method includes finding the optimal solution of a Nevanlinna-Pick interpolation problem with boundary conditions. A numerical example is also considered to show the effectiveness of the proposed method.

References

<table>
<thead>
<tr>
<th>Authors</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abbasov A</td>
<td>37</td>
</tr>
<tr>
<td>Abbasov M.E</td>
<td>40</td>
</tr>
<tr>
<td>Abdullayev S.H</td>
<td>276</td>
</tr>
<tr>
<td>Abdullayev V.M</td>
<td>1076</td>
</tr>
<tr>
<td>Ahmadov A.I</td>
<td>294</td>
</tr>
<tr>
<td>Ahmadoiva R.Y</td>
<td>355</td>
</tr>
<tr>
<td>Aida-Zade K.R</td>
<td>107, 110</td>
</tr>
<tr>
<td>Akbarov S.D</td>
<td>270</td>
</tr>
<tr>
<td>Akhundov A.A</td>
<td>43</td>
</tr>
<tr>
<td>Akhundov H.S</td>
<td>321</td>
</tr>
<tr>
<td>Akhundova E.M</td>
<td>43</td>
</tr>
<tr>
<td>Algiluyev R</td>
<td>46</td>
</tr>
<tr>
<td>Ali F</td>
<td>49</td>
</tr>
<tr>
<td>Ali I</td>
<td>49</td>
</tr>
<tr>
<td>Aliyev A.M</td>
<td>71</td>
</tr>
<tr>
<td>Aliyev F.A</td>
<td>52, 55, 58, 62, 204, 288</td>
</tr>
<tr>
<td>Aliyev S.A</td>
<td>65</td>
</tr>
<tr>
<td>Aliyev T.A</td>
<td>19</td>
</tr>
<tr>
<td>Aliyev T.M</td>
<td>68</td>
</tr>
<tr>
<td>Aliyev A.G</td>
<td>194</td>
</tr>
<tr>
<td>Aliyev N.A</td>
<td>52, 62, 71, 212, 285</td>
</tr>
<tr>
<td>Aliyev O.A</td>
<td>240</td>
</tr>
<tr>
<td>Aliyev R.M</td>
<td>346</td>
</tr>
<tr>
<td>Aliyeva T</td>
<td>37</td>
</tr>
<tr>
<td>Aliyeva T.H</td>
<td>83</td>
</tr>
<tr>
<td>Alizada T.A</td>
<td>19</td>
</tr>
<tr>
<td>Alkhateeb Areen</td>
<td>74</td>
</tr>
<tr>
<td>Allahvaranloo T</td>
<td>267</td>
</tr>
<tr>
<td>Allahverdi N</td>
<td>77</td>
</tr>
<tr>
<td>Amin R</td>
<td>116</td>
</tr>
<tr>
<td>Amiralii I</td>
<td>80</td>
</tr>
<tr>
<td>Amiraizay G.M</td>
<td>80</td>
</tr>
<tr>
<td>Amirova L.I</td>
<td>288</td>
</tr>
<tr>
<td>Amirova Sh.A</td>
<td>294</td>
</tr>
<tr>
<td>Ariazov G.T</td>
<td>83</td>
</tr>
<tr>
<td>Arcasey C.C</td>
<td>86</td>
</tr>
<tr>
<td>Asadov T.B</td>
<td>89</td>
</tr>
<tr>
<td>Asadzadeh M</td>
<td>92</td>
</tr>
<tr>
<td>Ashrafzadeh Y.R</td>
<td>110</td>
</tr>
<tr>
<td>Askercade E.N</td>
<td>98, 361</td>
</tr>
<tr>
<td>Aspressadeh E</td>
<td>95</td>
</tr>
<tr>
<td>Askberikey R.T</td>
<td>98</td>
</tr>
<tr>
<td>Asras S.S</td>
<td>364</td>
</tr>
<tr>
<td>Avci I</td>
<td>101</td>
</tr>
<tr>
<td>Avey A</td>
<td>104</td>
</tr>
<tr>
<td>Ayaz Y</td>
<td>113</td>
</tr>
<tr>
<td>Aziz I</td>
<td>116</td>
</tr>
<tr>
<td>Babakhani A</td>
<td>119</td>
</tr>
<tr>
<td>Babanti A.M</td>
<td>122</td>
</tr>
<tr>
<td>Baimamov E</td>
<td>113, 131</td>
</tr>
<tr>
<td>Bagirova G.A</td>
<td>65</td>
</tr>
<tr>
<td>Bas E</td>
<td>297</td>
</tr>
<tr>
<td>Baskurt K.B</td>
<td>336</td>
</tr>
<tr>
<td>Bawanex S</td>
<td>125</td>
</tr>
<tr>
<td>Bayramov V.A</td>
<td>346</td>
</tr>
<tr>
<td>Bazaka Yu.A</td>
<td>276</td>
</tr>
<tr>
<td>Bilaz S</td>
<td>68</td>
</tr>
<tr>
<td>Bilbul K.G</td>
<td>128</td>
</tr>
<tr>
<td>Bulat H</td>
<td>364</td>
</tr>
<tr>
<td>Castillio O</td>
<td>13</td>
</tr>
<tr>
<td>Cebesoy S</td>
<td>131</td>
</tr>
<tr>
<td>Colaner P</td>
<td>134</td>
</tr>
<tr>
<td>Dadashov Z.A</td>
<td>240</td>
</tr>
<tr>
<td>Demezre E.K</td>
<td>216</td>
</tr>
<tr>
<td>Dong Q</td>
<td>137</td>
</tr>
<tr>
<td>Dong S.-H</td>
<td>137</td>
</tr>
<tr>
<td>Dudin A.N</td>
<td>140</td>
</tr>
<tr>
<td>Dudin S.A</td>
<td>140</td>
</tr>
<tr>
<td>Dudina O.S</td>
<td>140</td>
</tr>
<tr>
<td>Dvalishvili Ph</td>
<td>143</td>
</tr>
<tr>
<td>Dvornikov M</td>
<td>146</td>
</tr>
<tr>
<td>Ebdopour Golanbar J</td>
<td>212</td>
</tr>
<tr>
<td>Edfendiev R</td>
<td>149</td>
</tr>
<tr>
<td>Edfendiyeva H.J</td>
<td>152</td>
</tr>
<tr>
<td>Emin S</td>
<td>155</td>
</tr>
<tr>
<td>Ercan A</td>
<td>297</td>
</tr>
<tr>
<td>Ergozi H</td>
<td>86</td>
</tr>
<tr>
<td>Ertosun N</td>
<td>77</td>
</tr>
<tr>
<td>Eskizmmmler S</td>
<td>158</td>
</tr>
<tr>
<td>Ezazit R</td>
<td>267</td>
</tr>
<tr>
<td>Farajdjeva Sh.A</td>
<td>288</td>
</tr>
<tr>
<td>Farhadova A.D</td>
<td>65</td>
</tr>
<tr>
<td>Fatuliyev F.T</td>
<td>46</td>
</tr>
<tr>
<td>Fazallahia M.A</td>
<td>161</td>
</tr>
<tr>
<td>Feyziev F.G</td>
<td>164</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Authors</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Feyzullayev Kh.A</td>
<td>167</td>
</tr>
<tr>
<td>Gamzaev Kh.M</td>
<td>170</td>
</tr>
<tr>
<td>Gasimova K.G</td>
<td>52, 204</td>
</tr>
<tr>
<td>Gerasymenko O.Yu</td>
<td>276</td>
</tr>
<tr>
<td>Geber B</td>
<td>182</td>
</tr>
<tr>
<td>Gorgodze N</td>
<td>173</td>
</tr>
<tr>
<td>Gulev G.A</td>
<td>19</td>
</tr>
<tr>
<td>Gurbanov F.I</td>
<td>176</td>
</tr>
<tr>
<td>Gurbanova R.S</td>
<td>179, 315</td>
</tr>
<tr>
<td>Gurbanova T.G</td>
<td>179</td>
</tr>
<tr>
<td>Gursoy M</td>
<td>158</td>
</tr>
<tr>
<td>Guzel M.S</td>
<td>361</td>
</tr>
<tr>
<td>Hajiyev A.A</td>
<td>315</td>
</tr>
<tr>
<td>Hajiyev N</td>
<td>279</td>
</tr>
<tr>
<td>Hajiyeva N.S</td>
<td>55,191</td>
</tr>
<tr>
<td>Hanaliloglu Z</td>
<td>182</td>
</tr>
<tr>
<td>Hasani Y.H</td>
<td>185, 279</td>
</tr>
<tr>
<td>Hasanov A.B</td>
<td>188</td>
</tr>
<tr>
<td>Hasanova G.K</td>
<td>206</td>
</tr>
<tr>
<td>Hashimov J.P</td>
<td>346</td>
</tr>
<tr>
<td>Huseynova N.Kh</td>
<td>240</td>
</tr>
<tr>
<td>Huseynova N.I</td>
<td>34</td>
</tr>
<tr>
<td>Huseynova N.Sh</td>
<td>55,191</td>
</tr>
<tr>
<td>Huseynov M.Z</td>
<td>340</td>
</tr>
<tr>
<td>Ibrahimov B</td>
<td>194</td>
</tr>
<tr>
<td>Ibrahimov B.G</td>
<td>122</td>
</tr>
<tr>
<td>Ibrahimov V.R</td>
<td>273</td>
</tr>
<tr>
<td>Imanova M.N</td>
<td>273</td>
</tr>
<tr>
<td>Inagaki T</td>
<td>198</td>
</tr>
<tr>
<td>Isayev M.M</td>
<td>201</td>
</tr>
<tr>
<td>Ismailov F.S</td>
<td>34</td>
</tr>
<tr>
<td>Ismayilov N.A</td>
<td>204</td>
</tr>
<tr>
<td>Iraz K</td>
<td>161</td>
</tr>
<tr>
<td>Jabbarov Sh</td>
<td>206</td>
</tr>
<tr>
<td>Jabrilyaeva Z.G</td>
<td>258</td>
</tr>
<tr>
<td>Jafarov R.G</td>
<td>146</td>
</tr>
<tr>
<td>Jahanshahi F</td>
<td>209</td>
</tr>
<tr>
<td>Jahanshahi M</td>
<td>209, 212, 216</td>
</tr>
<tr>
<td>Kabaniikin S.I</td>
<td>31, 219</td>
</tr>
<tr>
<td>Kars I</td>
<td>222</td>
</tr>
<tr>
<td>Kasimihely N</td>
<td>225</td>
</tr>
<tr>
<td>Kasimihely R</td>
<td>128, 225</td>
</tr>
<tr>
<td>Kayalar M</td>
<td>343</td>
</tr>
<tr>
<td>Khalilov M.S</td>
<td>167</td>
</tr>
<tr>
<td>Khan A</td>
<td>228</td>
</tr>
<tr>
<td>Khanyev T</td>
<td>182</td>
</tr>
<tr>
<td>Khankishiyev Z.F</td>
<td>231</td>
</tr>
<tr>
<td>Khelashvili A</td>
<td>234</td>
</tr>
<tr>
<td>Khudayarov B.H</td>
<td>240</td>
</tr>
<tr>
<td>Khusainov D.Ya</td>
<td>349</td>
</tr>
<tr>
<td>Klenkow Y.I</td>
<td>140</td>
</tr>
<tr>
<td>Konstantinov M</td>
<td>16</td>
</tr>
<tr>
<td>Koptubasi T</td>
<td>237</td>
</tr>
<tr>
<td>Kortuga Y.A.I</td>
<td>276</td>
</tr>
<tr>
<td>Kowalczuk P</td>
<td>92</td>
</tr>
<tr>
<td>Kryysi S.I</td>
<td>303, 306</td>
</tr>
<tr>
<td>Kuliev E.A</td>
<td>167</td>
</tr>
<tr>
<td>Kurbanov M.A</td>
<td>240</td>
</tr>
<tr>
<td>Larin V.B</td>
<td>58</td>
</tr>
<tr>
<td>Lakestani M</td>
<td>95</td>
</tr>
<tr>
<td>Maharramov I.M</td>
<td>204</td>
</tr>
<tr>
<td>Mahmudov E.N</td>
<td>243</td>
</tr>
<tr>
<td>Mahmudov N.I</td>
<td>246</td>
</tr>
<tr>
<td>Malik S</td>
<td>264</td>
</tr>
<tr>
<td>Malikov H.Kh</td>
<td>315</td>
</tr>
<tr>
<td>Mamadov J.G</td>
<td>249</td>
</tr>
<tr>
<td>Mamadov Sh</td>
<td>252</td>
</tr>
<tr>
<td>Mamadova N.G</td>
<td>176</td>
</tr>
<tr>
<td>Mammodov F</td>
<td>327</td>
</tr>
<tr>
<td>Mammodov G.H</td>
<td>164</td>
</tr>
<tr>
<td>Mammodov K.Sh</td>
<td>255</td>
</tr>
<tr>
<td>Mammodova M.A</td>
<td>315</td>
</tr>
<tr>
<td>Mammodova M.H</td>
<td>258</td>
</tr>
<tr>
<td>Mammodov N.N</td>
<td>62</td>
</tr>
<tr>
<td>Mammodova V.Y</td>
<td>255</td>
</tr>
<tr>
<td>Mansimov K.B</td>
<td>261</td>
</tr>
<tr>
<td>Mardanov M.C</td>
<td>264</td>
</tr>
<tr>
<td>Miahbadi Gholam A</td>
<td>267</td>
</tr>
<tr>
<td>Masaliev R.O</td>
<td>261</td>
</tr>
<tr>
<td>Mekitieve M.R</td>
<td>164</td>
</tr>
<tr>
<td>Melikov T</td>
<td>264</td>
</tr>
<tr>
<td>Mirzayev F.A</td>
<td>355</td>
</tr>
<tr>
<td>Mirzayeva H.G</td>
<td>321</td>
</tr>
<tr>
<td>Name</td>
<td>Page(s)</td>
</tr>
<tr>
<td>-----------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>Mukhin V.Ye.</td>
<td>276</td>
</tr>
<tr>
<td>Muradov A.</td>
<td>279</td>
</tr>
<tr>
<td>Musaev H.K.</td>
<td>282</td>
</tr>
<tr>
<td>Mustafayeva Y.Y.</td>
<td>285</td>
</tr>
<tr>
<td>Mutallimov M.M.</td>
<td>146, 288</td>
</tr>
<tr>
<td>Nadareishvili T.</td>
<td>234</td>
</tr>
<tr>
<td>Najafov A.M.</td>
<td>104</td>
</tr>
<tr>
<td>Nadirov U.</td>
<td>291</td>
</tr>
<tr>
<td>Nagiyev Sh.M.</td>
<td>294</td>
</tr>
<tr>
<td>Naimarov A.A.</td>
<td>55</td>
</tr>
<tr>
<td>Novikov N.S.</td>
<td>31, 219</td>
</tr>
<tr>
<td>Nutaliev A.F.</td>
<td>240</td>
</tr>
<tr>
<td>Nusratov O.G.</td>
<td>19</td>
</tr>
<tr>
<td>Onder U.G.</td>
<td>222</td>
</tr>
<tr>
<td>Orucova M.Sh.</td>
<td>191</td>
</tr>
<tr>
<td>Ozbay H.</td>
<td>379</td>
</tr>
<tr>
<td>Ozbay G.</td>
<td>113</td>
</tr>
<tr>
<td>Panakhov E.S.</td>
<td>297, 343</td>
</tr>
<tr>
<td>Pashayev A.B.</td>
<td>300</td>
</tr>
<tr>
<td>Pashayev F.H.</td>
<td>19</td>
</tr>
<tr>
<td>Petkov P.</td>
<td>16</td>
</tr>
<tr>
<td>Pogoriliy S.D.</td>
<td>303, 306</td>
</tr>
<tr>
<td>Poladova A.</td>
<td>182</td>
</tr>
<tr>
<td>Polat R.</td>
<td>158</td>
</tr>
<tr>
<td>Rajabov M.F.</td>
<td>62</td>
</tr>
<tr>
<td>Rahimov A.B.</td>
<td>309</td>
</tr>
<tr>
<td>Rahimov F.H.</td>
<td>65</td>
</tr>
<tr>
<td>Ramazanov A.B.</td>
<td>312</td>
</tr>
<tr>
<td>Ramazanova E.E.</td>
<td>315</td>
</tr>
<tr>
<td>Ramazanova I.S.</td>
<td>240</td>
</tr>
<tr>
<td>Rustamova L.A.</td>
<td>152</td>
</tr>
<tr>
<td>Ruzhansky M.</td>
<td>25</td>
</tr>
<tr>
<td>Rzaev A.G.</td>
<td>19</td>
</tr>
<tr>
<td>Rzayeva N.E.</td>
<td>19</td>
</tr>
<tr>
<td>Sabzaliev M.M.</td>
<td>318</td>
</tr>
<tr>
<td>Sabzalieva I.M.</td>
<td>318</td>
</tr>
<tr>
<td>Sabzev E.N.</td>
<td>300</td>
</tr>
<tr>
<td>Sadigov E.N.</td>
<td>188</td>
</tr>
<tr>
<td>Sadigov M.A.</td>
<td>321, 324</td>
</tr>
<tr>
<td>Safradova N.A.</td>
<td>52, 55, 191</td>
</tr>
<tr>
<td>Salavatov T.Sh.</td>
<td>327</td>
</tr>
<tr>
<td>Samedov Z.A.</td>
<td>330</td>
</tr>
<tr>
<td>Samet R.</td>
<td>333, 336</td>
</tr>
<tr>
<td>Samreec S.</td>
<td>340</td>
</tr>
<tr>
<td>Sarac T.</td>
<td>225</td>
</tr>
<tr>
<td>Sarfraz M.</td>
<td>28, 340</td>
</tr>
<tr>
<td>Sat M.</td>
<td>343</td>
</tr>
<tr>
<td>Sattarova U.E.</td>
<td>19</td>
</tr>
<tr>
<td>Shabanov S.A.</td>
<td>185</td>
</tr>
<tr>
<td>Shafizade E.R.</td>
<td>346</td>
</tr>
<tr>
<td>Shatyrko A.V.</td>
<td>349</td>
</tr>
<tr>
<td>Shavadze T.</td>
<td>352</td>
</tr>
<tr>
<td>Shahkimskaya R.Y.</td>
<td>355</td>
</tr>
<tr>
<td>Shaleshin M.A.</td>
<td>31, 219</td>
</tr>
<tr>
<td>Siraj I.U.</td>
<td>385</td>
</tr>
<tr>
<td>Slyndo M.S.</td>
<td>303, 306</td>
</tr>
<tr>
<td>Soldmar S.</td>
<td>131</td>
</tr>
<tr>
<td>Soylu B.E.</td>
<td>361</td>
</tr>
<tr>
<td>Standar C.</td>
<td>92</td>
</tr>
<tr>
<td>Sulaiman T.A.</td>
<td>364</td>
</tr>
<tr>
<td>Sukizmanov B.A.</td>
<td>24</td>
</tr>
<tr>
<td>Sun G.H.</td>
<td>137</td>
</tr>
<tr>
<td>Tadumadze T.</td>
<td>367</td>
</tr>
<tr>
<td>Taghiyeva Sh.</td>
<td>252</td>
</tr>
<tr>
<td>Tagiev R.M.</td>
<td>62</td>
</tr>
<tr>
<td>Tagi-Zadeh A.T.</td>
<td>370</td>
</tr>
<tr>
<td>Torres-Arenas A.J.</td>
<td>137</td>
</tr>
<tr>
<td>Valizada A.H.</td>
<td>146</td>
</tr>
<tr>
<td>Veliev E.F.</td>
<td>34</td>
</tr>
<tr>
<td>Velieva N.I.</td>
<td>373</td>
</tr>
<tr>
<td>Yang K.Ch.</td>
<td>68</td>
</tr>
<tr>
<td>Yagubov M.H.</td>
<td>376</td>
</tr>
<tr>
<td>Yaman A.U.</td>
<td>333</td>
</tr>
<tr>
<td>Yasmeen Sh.</td>
<td>116</td>
</tr>
<tr>
<td>Ye! G.</td>
<td>364</td>
</tr>
<tr>
<td>Yorgancioglu Z.O.</td>
<td>158</td>
</tr>
<tr>
<td>Yucesoy V.</td>
<td>379</td>
</tr>
<tr>
<td>Yusubov Sh.Sh.</td>
<td>376</td>
</tr>
<tr>
<td>Zaman G.</td>
<td>228</td>
</tr>
<tr>
<td>Zaaraa E.</td>
<td>22</td>
</tr>
</tbody>
</table>